GENERALIZED KUMMER CONGRUENCES FOR PRODUCTS OF SEQUENCES

By Harlan Stevens

1. **Introduction.** If p is any fixed rational prime, then in this paper we shall always let R_p denote the set of rational numbers that are integral (mod p).

Let $\{a_n\}$ be a sequence of numbers in R_p that satisfy the congruence

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} \lambda_1^{r-s} a(m + s(p^{t_1} - 1)) \equiv 0 \pmod{p^r} \tag{1.1}$$

for all $m \geq r \geq 1$ and for some positive integer t_1. The multiplier λ_1 is also in R_p and $a_n = a(n)$. We shall call (1.1) Kummer's congruence for $\{a_n\}$. For example, Nielsen [8, Chapter 14] shows that in case $p > 2$, $t_1 = 1$ and $\lambda_1 = 1$, formula (1.1) holds for $a_n = E_n$, the Euler numbers in the even suffix notation. If $\{b_n\}$ is a second sequence of numbers in R_p that satisfy

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} \lambda_2^{r-s} b(m + s(p^{t_2} - 1)) \equiv 0 \pmod{p^r} \tag{1.2}$$

for all $m \geq r \geq 1$ and for some positive integer t_2, then a natural way to form a composition sequence is by means of the Hurwitz product. Put

$$c_n = \sum_{j=0}^{n} \binom{n}{j} a_j b_{n-j} \qquad (n = 0, 1, 2, \cdots). \tag{1.3}$$

We shall call the sequence $\{c_n\}$ the Hurwitz product of the sequences $\{a_n\}$ and $\{b_n\}$. In the special case $t_1 = t_2 = 1$, Carlitz [5] has proved the following result: if $\lambda_1\lambda_2 \not\equiv 0$ (mod p) and t is the least positive integer such that

$$\lambda_1^t \equiv \lambda_2^t \equiv k \pmod{p}$$

for some k, then

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} k^{r-s} c(m + s(p^t - 1)) \equiv 0 \pmod{p^r} \tag{1.4}$$

for all $m \geq r \geq 1$, where $\{c_n\}$ is defined by (1.3). More generally he has shown that

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} k^{p^z(r-s)} c(m + sp^z(p^t - 1)) \equiv 0 \pmod{p^{rz+r}} \tag{1.5}$$

for $r \geq 1$, $z \geq 0$, $m \geq rz + r$. In case $\lambda_1 \equiv 0$ (mod p) and $\lambda_2 \not\equiv 0$ (mod p), these results do not hold and another congruence [5, Theorem 4] is obtained.

These results raise other interesting questions. We may seek, for example,

Received May 11, 1960. Research supported by National Science Foundation grant G-9425.