LINKS OF BRUNNIAN TYPE

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1. Introduction. We consider in this paper links of multiplicity n ($n > 1$) with the property that they are unsplittable but every proper sublink is completely splittable. We say such links are of Brunnian (splitting) type, since examples, for arbitrary n, were first given by H. Brunn [3]. In this early contribution to knot theory Brunn investigated the existence of links with prescribed splitting properties; his considerations were based on links of Brunnian type. But, as Brunn remarked, "a mathematically rigorous formulation, applicable to every case, of the criteria whether two or more rings are inseparably interlocked is lacking up to date, $\cdots$, hence the investigations of this paper are partially based on empiricism only". Since then the methods of knot theory have widely overcome this lack, but to our best knowledge the unsplittability has not been proved rigorously for any of the links constructed by Brunn. Such a proof is given in §4 by using a geometrical lemma (§2) which has some interest in itself. In §§5 and 6 we are concerned with some related questions: additional properties of Brunn's example, another geometrically simple and highly symmetrical n-link of Brunnian type and, finally, the construction (for every n and k such that $1 < k \leq n$) of an n-link with the property that no sublink of k or more components is splittable but every sublink consisting of less than k components is completely splittable.

2. Complete splitting of links. A *link* L of multiplicity n is considered here as the image set of a homeomorphism $\mathfrak{L}: C_n \to R^3$, where C_n is a family of n disjoint circles and R^3 the Euclidean 3-space. We restrict our attention to *tame* links L (and spheres), i.e., for some homeomorphism $\Phi : R^3 \to R^3$ the image set $\phi(L)$ is a polygonal link (a polyhedral sphere, respectively). A tame n-link L is said to be *splittable* if there exists in R^3 a tame 2-sphere S such that $L \cap S = 0$, $L \cap \text{int } S \neq 0$, $L \cap \text{ext } S \neq 0$. The n-link L with components $K_1, \cdots, K_n$ is said to be *completely splittable* if there exist n disjoint tame 2-spheres $S_1, \cdots, S_n$ such that $K_i \subset \text{int } S_i \subset \text{ext } S_k$ ($i, k = 1, \cdots, n$; $k \neq i$).

Lemma 1. *Let L be a tame link with components $K_1, \cdots, K_n$ ($n > 1$). If L is splittable and $L - K_i$, for $i = 1, \cdots, n$, is completely splittable, then L is completely splittable.*

Proof. By [2] and [9] we need only consider polyhedral spheres and links. Let L be a splittable n-link. So we may assume that the 2-sphere S contains

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