STRUCTURE OF MEASURABLE TRANSFORMATIONS

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A property of measurable transformations may be called *localizable* if, for any transformation T, there is an invariant set A so that T regarded as a transformation on A has the property, and so that T regarded as a transformation on the complement of A does not have the property. An example of a localizable property for invertible transformations is incompressibility [4; 46] and [3; 12]. In the latter reference, the question of the localizability of incompressibility is raised for transformations which have no inverse. In this note we show that each of the following is a localizable property: measure preserving; existence of equivalent invariant measures; incompressibility; absolute continuity.

Throughout, let $(X, \mathcal{S}, m)$ denote a totally finite measure space. Unless otherwise qualified, T will denote an arbitrary measurable transformation of the space. All sets chosen in *a priori* manner are measurable.

A set E is said to be *invariant* under T if $T^{-1}E = E$. Let $\mathcal{A}$ denote the family of all invariant sets; $\mathcal{A}$ is clearly a Boolean σ-algebra of sets. We may therefore consider the totally finite measure space $(X, \mathcal{A}, m)$. Let $\mathcal{K}$ be any σ-ideal of $\mathcal{A}$ which contains all invariant null sets. Choose in $\mathcal{K}$ a maximal disjoint family $\{K_i\}$ of sets of positive measure. Since $m(X) < \infty$, then $\sum_i m(K_i) < \infty$, and hence the collection must be countable. Let $K = \bigcup_i K_i$; since $\mathcal{K}$ is a σ-ideal, $K \in \mathcal{K}$. A set $A \in \mathcal{A}$ belongs to $\mathcal{K}$ if and only if $m(A - K) = 0$. For if this condition is satisfied, then $A - K \in \mathcal{K}$, since $\mathcal{K}$ contains all invariant null sets, and $A \cap K \in \mathcal{K}$ since $\mathcal{K}$ is an ideal; thus $A \in \mathcal{K}$. Conversely, if $A \in \mathcal{K}$, then $m(A - K_i) = 0$, because of the maximality of the family $\{K_i\}$. Then $m(A - K) = 0$. For convenience, we will call a σ-ideal in $\mathcal{A}$ which contains all invariant null sets a *full* σ-ideal. A set K which determines a full σ-ideal in $\mathcal{A}$ in the above manner will be called a *localizing set* for the ideal. If K_1, K_2 are two localizing sets for the same full σ-ideal, then $m(K_1 + K_2) = 0$, where $+$ denotes symmetric difference. Any invariant set is a localizing for a full σ-ideal in $\mathcal{A}$. (See [6, Theorem 5.1] for the most general setting of this comment.)

A transformation is called *incompressible* (or *conservative*) if and only if $m(T^{-1}E - E) = 0$ whenever $E \subset T^{-1}E$. An invariant set A will be called a *domain of incompressibility* for T if T is incompressible as a transformation on A.

Theorem 1. *The domains of incompressibility of T form a full σ-ideal of $\mathcal{A}$.*

Proof. Let $\mathcal{K}$ denote the set of all domains of incompressibility of T. Let A be an invariant null set, and suppose $E \subset A$ and $E \subset T^{-1}E$. Since A is invariant, this implies that $T^{-1}E \subset A$; hence $m(T^{-1}E) = 0$ and $m(T^{-1}E - E) = 0$. Therefore $\mathcal{K}$ contains all invariant null sets. Next, let $K \in \mathcal{K}$, $A \in \mathcal{A}$, $A \subset K$. If $E \subset A$, $E \subset T^{-1}E$, then $E \subset T^{-1}E \subset A \subset K$, because of the invariance of

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