THE VANISHING OF THE HOMOGENEOUS PRODUCT SUM ON THREE LETTERS

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1. **Statement of result.** By the homogeneous product sum $H_n = H_n(x, y, z)$ on three letters x, y and z we understand the sum of all the symmetric functions of x, y and z of weight n. Thus $H_1 = x + y + z$, $H_2 = x^2 + y^2 + z^2 + xy + yz + zx$, $H_3 = x^3 + y^3 + z^3 + x^2y + x^2z + y^2x + y^2z + z^2x + z^2y + xyz$ and so on. It is an important question in the theory of integral cubic recurrences to decide whether the Diophantine equation $H_n = 0$, $n > 1$, can have non-trivial solutions. (Ward [1], [3]). In a recent paper in this journal (Ward [2]) referred to hereafter as D.P., it was shown that this can happen only when n is odd. It was also proved by descent that $H_3 = 0$ has only trivial solutions. I prove here the following generalization.

THEOREM 1.1. *The Diophantine equation*

$$H_n(x, y, z) = 0 \tag{1}$$

has only trivial solutions whenever $n + 2$ is a prime number greater than three.

This result makes the conjecture that (1.1) has only trivial solutions for all values of n greater than one considerably more plausible (Ward [1]). The remainder of this paper is devoted to a proof by contradiction of Theorem 1.1.

2. **Preliminary reductions.** Throughout the remainder of this paper, p denotes a fixed prime greater than three. By a trivial solution of (1.1) we understood one in which $xyz = 0$. Assume that there exists a non-trivial solution $x = a$, $y = b$, $z = c$. Then from results given in D.P. we are entitled to assume

$$H_n(a, b, c) = 0, \qquad n + 2 = p > 3. \tag{2.1}$$

$$abc \neq 0. \tag{2.2}$$

$$(a, b) = (b, c) = (c, a) = 1. \tag{2.3}$$

We shall deduce a contradiction from these assumptions.

3. We begin with some simple lemmas.

LEMMA 3.1. *Under the hypotheses of §2, there exist two co-prime non-zero integers N and M such that*

$$\frac{a^p - b^p}{a - b} = \frac{b^p - c^p}{b - c} = \frac{c^p - a^p}{c - a} = N. \tag{3.1}$$

$$ab\left(\frac{a^{p-1} - b^{p-1}}{a - b}\right) = bc\left(\frac{b^{p-1} - c^{p-1}}{b - c}\right) = ca\left(\frac{c^{p-1} - a^{p-1}}{c - a}\right) = M. \tag{3.2}$$

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