DECOMPOSITION OF LINEAR FUNCTIONALS ON RIESZ SPACES

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Preliminaries. Let V be a vector space over the real numbers. If we introduce a multiplication on V, and thus make V an algebra, the multiplicative linear functionals (i.e., linear functionals F such that $F(fg) = Ff \cdot Fg$) on V are of special interest. On the other hand, we may introduce an order structure on V, and thus make V a Riesz space (i.e., a vector lattice). It is natural to suppose that in the latter case there is an important class of linear functionals corresponding to the multiplicative ones in the former. We shall define such a class of linear functionals and study their characteristic properties. Using these functionals, we shall prove that any positive linear functional (i.e., $Ff \geq 0$ whenever $f \geq 0$) on a Riesz space may be decomposed into an "atomic" and a "diffuse" part.

An important special case is the case where the elements of V are real-valued functions on a set E. We suppose that the linear operations on V are so defined that $(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha[f(x)]$ whenever $f, g \in V$ and α is a real number. We shall call V a Riesz space of functions on E if it contains with f, g the function $f \vee g$ defined by $(f \vee g)(x) = \max [f(x), g(x)]$. It is clear that such a V is a lattice with the ordering $f \geq g$ if $f(x) \geq g(x)$ for all x. We shall call V an algebra of functions on E if it contains with f, g the function fg defined by $(fg)(x) = f(x)g(x)$.

Suppose now that V is a Riesz space of bounded functions on E, in the sense specified above, and that V contains the constant functions. For convenience, suppose V separates points of E. In this still more special case, we may reason as follows. Let $\bar{V}$ be the uniform closure of V in the space of all bounded functions on E. Clearly $\bar{V}$ is a Riesz space of functions on E. We show $\bar{V}$ is an algebra. Consider any $f \in \bar{V}$ with $0 \leq f \leq 1$. Define real-valued functions $\varphi_1(\alpha), \varphi_2(\alpha), \cdots$ on the closed interval [0, 1] as follows: Let $\varphi_n(\alpha)$ be the function defined on [0, 1] which is equal to α^2 when $\alpha = 0, 1/n, 2/n, \cdots, 1$ and is linear between these values. We note that φ_n may be obtained from linear functions by the lattice operation "$\vee$". Therefore the composite functions $\varphi_n(f)$ are in V. Since $\varphi_n(\alpha) \to \alpha^2$ uniformly on [0, 1], $\varphi_n(f) \to f^2$ uniformly. Hence $f^2 \in \bar{V}$. It follows that $\bar{V}$ is an algebra. (We note that we have used the hypotheses that V contain the constants and that the functions in V be bounded.) Each positive linear functional F on V is continuous in the uniform topology since $1 \in V$. Thus, by the Hahn-Banach theorem, we may extend F to be a continuous functional on $\bar{V}$. Since V is dense in $\bar{V}$, the extension is unique. The extended F is clearly positive since it is positive on V, which is dense in $\bar{V}$. It

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