ASYMPTOTIC EXPANSION OF DOUBLE FOURIER TRANSFORMS

By R. J. Duffin and D. H. Shaffer

1. **Introduction.** This paper is concerned with the effect of point singularities of a function on the asymptotic expansion of its Fourier transform at infinity. For functions $F(x)$ of a single variable, this problem has received much study, and reference may be made to the works of Titchmarsh [9; Theorems 126 and 127] and Lighthill [6; Chapter 4]. In this paper attention is confined to functions $F(x, y)$ of two variables. The asymptotic expansion of the Fourier transform is found for certain types of singularities termed "simple".

The problem of the asymptotic behaviour of the Fourier transform of a function $F(x, y, z)$ of three variables has been treated by one of us in a previous paper [2]. The present treatment is patterned after the previous treatment, but goes further in several directions. It is worth noting that, in certain respects, the problem for functions of two variables is more difficult to analyse than the case of three variables.

The formulae developed in this paper have application to partial difference equations. For example, in discrete potential theory the Green's function g is expressible as a double Fourier transform. ($2\pi g$ is analogous to the function $\log r$ in continuous potential theory.) Several terms in the asymptotic expansion of g are calculated below. This expansion has been quoted and employed in two papers in anticipation of the proof supplied here [3], [5].

2. **Definitions.** The double Fourier transform of a function $F(x, y)$ is defined as

$$f(a, b) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(ax+by)} F(x, y)\, dx\, dy. \tag{1}$$

Here, a and b denote continuous real variables. Relation (1) is abbreviated as $f = T(F)$. Also the following notation is used: $r = (x^2 + y^2)^{\frac{1}{2}}$, $k = (a^2 + b^2)^{\frac{1}{2}}$, $k \cdot r = ax + by$, and $\iint - dA$ for an integral over the entire plane.

Of concern are functions which are expressible in the form

$$F(x, y) = r^q x^\alpha y^\beta + F_1(x, y). \tag{2}$$

Here, $F_1(x, y)$ is a function with partial derivatives of all orders, q is a real constant, and α and β are positive integers or zero. If q is not zero, or an even positive integer, then F is said to have a "simple singularity" at the origin.

If $F(x, y)$ is defined by (2), then the function $F_t = F(x - x_0\,, y - y_0)$ may be said to have a simple singularity at the point $(x_0\,, y_0)$. Of course the Fourier

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