POSITIVE DEFINITE FUNCTIONS AND INDUCED REPRESENTATIONS

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1. **Introduction.** It is well known that if φ is a continuous positive definite function on a locally compact group G, then φ arises from a unitary representation U of G, and an element h in the Hilbert space of U, by way of the identity

$$\varphi(\sigma) = (U_\sigma h, h),$$

and it is easy to see that every function defined this way is positive definite.

If $\mathcal{E}$ is a family of projections on H forming a complete Boolean algebra in the lattice of all projections, and if $E \rightarrow [E]$ is a σ-homomorphism of a σ-algebra of subsets of a space S onto $\mathcal{E}$, then a measure can be defined on S by

$$\mu(E) = \| [E]h \|^2 = (h, [E]h). \tag{1}$$

If $\mathcal{E}$ reduces U, then it is known that φ has a *direct integral decomposition* over $\mathcal{E}$, expressing the various associated "disjoint parts" of φ, $\varphi_E(\sigma) = (U_\sigma h, [E]h)$, as the integral over the measurable set E of "fiber" positive definite functions. That is

$$\varphi_E(\sigma) = (U_\sigma h, [E]h) = \int_E \varphi(\sigma, s)\, d\mu(s),$$

where φ is a measurable function of $\langle \sigma, s \rangle$ and is positive definite in σ for almost all s.

Now these "fiber" positive definite functions are associated with "fiber" unitary representations of G, and the above direct integral decomposition of φ expresses the fact that the unitary representation U is reduced by the algebra $\mathcal{E}$ to a direct integral of "fiber" representations. It is possible to arrive at the direct integral decomposition of φ by first proving the direct integral decomposition theorem for U. This requires the full machinery of the von Neumann direct integral theory for Hilbert spaces and operator algebras on them. However, it was noticed independently by Godement and Segal [1], [5] that the direct integral decomposition of φ can be given an extremely simple direct proof, and that this in turn implies the direct integral decomposition of U. Because of this the decomposition theory of an operator algebra arising from a group representation can be approached in a much simpler fashion than can the general theory.

In this paper we shall prove directly a new direct integral decomposition theorem for φ which is related in a similar way to the direct-integral-like structure

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