WIDDER'S INVERSION FORMULA FOR THE LAMBERT TRANSFORM

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1. The Lambert transform $F(s)$ of a function $a(t)$ (assumed to be Lebesgue integrable on $[0, x]$ for every $x > 0$) is defined formally by the equation

$$F(s) = \int_0^\infty \frac{ta(t)}{e^{st} - 1}\, dt. \tag{1}$$

In 1950 Widder [5, Theorems 3.2 and 3.3] obtained two inversion formulae for the transform $F(s)$:

Theorem 1. *Suppose that*

$$\int_0^\infty | a(t) |\, dt < \infty, \tag{2}$$

that for some $\delta > 0$

$$\lim_{t \to 0+} a(t) t^{1-\delta} = 0, \tag{3}$$

and that $F(s)$ *is defined by the integral* (1) (*which certainly converges for every* $s > 0$). *Then if* $x > 0$ *and* $a(t)$ *is continuous at* $t = x$,

$$xa(x) = \lim_{k \to \infty} \frac{(-1)^k}{k!} \left(\frac{k}{x}\right)^{k+1} \sum_{n=1}^\infty \mu(n) n^k F^{(k)}\left(\frac{nk}{x}\right),$$

where $\mu(n)$ *is the Möbius function and* $F^{(k)}(s) = d^k/ds^k\, F(s)$.

Theorem 2. *Under the hypotheses of Theorem* 1, *if* $x > 0$ *and* $a(t)$ *is continuous at* $t = x, \frac{1}{2}x, \frac{1}{3}x, \cdots$, *then*

$$xa(x) = \sum_{n=1}^\infty \mu(n) \lim_{k \to \infty} \frac{(-1)^k}{k!} \left(\frac{k}{x}\right)^{k+1} n^k F^{(k)}\left(\frac{nk}{x}\right).$$

The object of the present paper is to obtain an inversion formula for (1) without using the conditions (2) and (3). We obtain such a formula under the hypothesis that for some $s_0 > 0$ the integral

$$\int_0^\infty \frac{ta(t)}{e^{s_0 t} - 1}\, dt$$

is convergent. It can be shown (see Widder [5], Theorems 1.1 and 1.2) that this hypothesis implies that the integral (1) converges for all $s \geq s_0$. For the sake

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