BOUNDS FOR THE MAXIMAL CHARACTERISTIC ROOT OF A NON-NEGATIVE IRREDUCIBLE MATRIX

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1. Let A be an $n \times n$ non-negative irreducible matrix with row sums (all summations go from 1 to n.)

$$(1) \qquad r_\nu = \sum_\mu a_{\nu\mu}\,, \qquad R = \max_\nu r_\nu\,, \qquad r = \min_\nu r_\nu\,.$$

We shall suppose throughout that

$$(2) \qquad r < R.$$

It is well known that because of (2) the maximal characteristic root of A satisfies the inequality

$$(3) \qquad r < \omega < R.$$

In §2 we shall use simple arguments to determine bounds L and U for ω satisfying

$$(4) \qquad r < L \leq \omega \leq U < R,$$

which may be computed easily in terms of the elements of A; more precisely, L and U will depend only on r and R in (1) and

$$(5) \qquad \rho = \frac{1}{n} \sum_\nu r_\nu$$

$$(6) \qquad \lambda = \min_\nu a_{\nu\nu}$$

$$(7) \qquad \kappa = \min_{\nu \neq \mu} a_{\nu\mu}\ (a_{\nu\mu} > 0),$$

i.e. κ is the minimum of the non-vanishing $a_{\nu\mu}$ with $\nu \neq \mu$.

In §3 a more refined and longer argument will lead to better bounds which still depend only on the r_ν in (1) and κ and λ, but require more computation.

When A is positive, bounds satisfying the inequality (4) have already been found by Ledermann [2] and improved by Ostrowski [3] and Brauer [1], but these bounds may coincide with r and R if A has zero elements. Of course, there are bounds which for many matrices A are better than r or R, but these may again reduce to r and R in some cases. For example, one of us has proved, [4], that $\omega \leq \max_\nu r_\nu^p z_\nu^{1-p}$ where $z_\nu = \sum_\mu a_{\mu\nu}$ and $0 \leq p \leq 1$, but this bound equals R if A is symmetric.

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