MULTIPLICATION FORMULAS FOR GENERALIZED BERNOULLI AND EULER POLYNOMIALS

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1. Put

$$\frac{te^{xt}}{e^t - 1} = \sum_0^\infty B_n(x)\frac{t^n}{n!}, \qquad \frac{2e^{xt}}{e^t + 1} = \sum_0^\infty E_n(x)\frac{t^n}{n!}.$$

It is familiar that

$$\sum_{r=0}^{k} B_m\left(x + \frac{r}{k}\right) = k^{1-m}B_m(kx), \tag{1.1}$$

$$\sum_{r=0}^{k} (-1)^r E_m\left(x + \frac{r}{k}\right) = k^{-m}E_m(kx) \qquad (k \quad \text{odd}), \tag{1.2}$$

$$\sum_{r=0}^{k} (-1)^r B_m\left(x + \frac{r}{k}\right) = -\tfrac{1}{2}mk^{1-m}E_{m-1}(kx) \qquad (k \quad \text{even}). \tag{1.3}$$

Also, as Nielsen [5; 54] has pointed out, if a normalized polynomial satisfies (1.1) for a single value of $k > 1$, then it is identical with $B_m(x)$; similarily if a normalized polynomial satisfies (1.2) for a single odd value of $k > 1$, then it is identical with $E_m(x)$.

If we define the functions $\bar{B}_m(x)$, $\bar{E}_m(x)$ by means of

$$\bar{B}_m(x) = B_m(x) \qquad (0 \le x < 1), \qquad \bar{B}_m(x+1) = \bar{B}_m(x),$$

$$\bar{E}_m(x) = E_m(x) \qquad (0 \le x < 1), \qquad \bar{E}_m(x+1) = -\bar{E}_m(x) \qquad (m \ge 1),$$

then it is easily seen that (1.1), (1.2), (1.3) hold for $\bar{B}_m(x)$, $\bar{E}_m(x)$ also.

In a recent paper [1], the writer has obtained, using a method employed by Mordell [4] in extending some work of Mikolás [3], the following results.

Let $a_1, \cdots, a_n$ be positive integers that are relatively prime in pairs and let $A = a_1a_2 \cdots a_n$. Then if k is an arbitrary integer ≥ 1, we have

$$\sum_{r=0}^{kA-1} \bar{B}_{m_1}\left(x_1 + \frac{r}{ka_1}\right) \cdots \bar{B}_{m_n}\left(x_n + \frac{r}{ka_n}\right) = C \sum_{r=0}^{k-1} \bar{B}_{m_1}\left(a_1x_1 + \frac{r}{k}\right) \cdots \bar{B}_{m_n}\left(a_1x_1 + \frac{r}{k}\right), \tag{1.4}$$

where

$$C = a_1^{1-m_1}a_2^{1-m_2} \cdots a_n^{1-m_n}. \tag{1.5}$$

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