SOME EMBEDDINGS, RECURRENCE PROPERTIES, AND THE BIRKHOFF-MARKOV THEOREM FOR TRANSFORMATION GROUPS

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1. **Introduction.** We consider the topological transformation group (X, T, π) or more briefly (X, T), where X is a compact Hausdorff space and T is a generative group; that is, T is Abelian and is generated by a compact neighborhood of the identity. The classical Birkhoff-Markov theorem [3], [4; 49, 81] concerns itself with the case where T is the additive group of the reals and states that the center is a center of attraction. (The author is indebted to Professor Hedlund who recently pointed out Bernard's solution [1] to the problem.) Bernard [1] has treated a form of this problem for generative groups and has found a solution. His approach to the problem is to take limits over various simply-ordered sets of neighborhoods and to compare them. The approach used here is much more naïve. Since T is generative, it is isomorphic to $K \times \mathcal{I}^q \times \mathcal{R}^p$, where K is compact Abelian, $\mathcal{I}$ and $\mathcal{R}$ are the additive groups of the integers and the reals, and p and q are non-negative integers [4; 110]. The problem is first solved for $T = \mathcal{R}^p$, and then we reduce the general problem to this. This approach has the following advantages: in the case where $T = \mathcal{R}^p$ the limits encountered are more feasibly calculated, and some information is obtained on the structure of replete semigroups (§3); in §4 the relation between compact subgroups and recurrence is examined; and in §5 a known embedding technique is generalized.

2. **Centers of attraction.** Let (X, T) be a locally compact transformation group of the compact Hausdorff space X (T acting on the right). Let μ be a left invariant Haar measure on T and define the following function $f: T \to \{0, 1\}$. If $x \in X$ and $U \subset X$, then

$$f(t; x, U) = \begin{matrix} 1 & xt \in U \\ 0 & xt \notin U. \end{matrix} \qquad (1.1)$$

It is clear that if U is a Borel set, then f is Borel measurable in t for each $x \in X$. Henceforth we will always assume that f is μ measurable. If $S \subset T$ is a μ measurable subset of positive finite measure, we define

$$Q(x, U, S) = \frac{\int_S f(t; x, U)\, d\mu}{\mu(S)}.$$

The above expression is independent of the particular left invariant Haar measure chosen.

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