NATURAL SUMS AND DIRECT DECOMPOSITIONS

By J. R. Isbell

Introduction. A. G. Kurosh has introduced a novel notion of addition of mappings in non-Abelian categories [3]. Basically it is Fitting's addition of homomorphisms with commuting values [1], and thus a direct generalization of addition in Abelian categories; but there are three striking peculiarities in Kurosh's treatment. Much the most striking is that after giving a perfectly self-dual description of addition, he applies it to a study of direct decompositions in categories in which the notion of direct decomposition is not self-dual. Secondly, he allows infinite sums, so that the operation is new even in the Abelian case. The third peculiarity is associated with the first two; Kurosh introduces addition purely axiomatically, and does not suggest an interpretation even for infinite sums in Abelian groups.

The purpose of this paper is to describe an addition in a rather restricted class of categories, which satisfies all but one of Kurosh's axioms and all of his conclusions from the axioms. It is associated with a product between the free product and the direct product, which we call the *weak product*. In Abelian categories, weak product = free product. In the category of all groups, finite weak product = finite direct product; the infinite weak product is something different. In topological spaces with base point, finite weak product = finite free (wedge) product.

The categories are required to have a two-sided zero, free products, and direct products, and to satisfy Grothendieck's axioms AB 1 and AB 2 [2]: Every mapping has a kernel and cokernel, and coimages are isomorphic with images. Thus almost all of the apparatus of an Abelian category is needed. As a matter of fact, this treatment does not cover non-compact spaces with base point (for coimages are not isomorphic with images). The first half of the paper presents a special treatment of finite sums which covers this case and, more to the point, is simpler.

The weak product is the image of the free product under the natural mapping of the free product into the direct product. In particular, for finite products in groups, the weak product coincides with the direct product, and Kurosh's work is at least a proper generalization of Fitting's. I do not know if there is an addition satisfying Kurosh's axioms which is related to infinite direct decompositions of groups.

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1. **Some special cases.** Terminology not explained here is from Chapter I

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