DETERMINATION OF CERTAIN ROOTS OF UNITY IN THE THEORY OF AUTOMORPHIC FORMS OF DIMENSION ZERO

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1. **Introduction.** In [1] Hecke investigated the class of properly discontinuous groups $\{G(\lambda_q)\}$, where $G(\lambda_q)$ is generated by the two substitutions $S(\tau) = \tau + \lambda_q$, $T(\tau) = -1/\tau$. Here, $\lambda_q = 2 \cos \pi/q$, where q is an integer ≥ 3. For $q = 3$ we have the modular group, while for $q = 4, 6$ we obtain $\lambda_q = \sqrt{2}, \sqrt{3}$ respectively and we denote the corresponding groups by $G(\sqrt{2})$ and $G(\sqrt{3})$. The groups $G(\sqrt{2})$ and $G(\sqrt{3})$ are completely known and, to my knowledge, these two groups and the modular group are the only Hecke groups for which this is true, although some progress has been made by Rosen [7] for the entire class $\{G(\lambda_q)\}$.

In [5] Raleigh computes the Fourier series of the invariants for the groups $G(\sqrt{2})$ and $G(\sqrt{3})$ by making use of the circle method, while in [3] the author constructs forms of all nonnegative even integral dimensions, with multiplier system identically one, for these groups. In order to carry over the methods of [3] to dimension zero, with *arbitrary* multiplier systems, it is necessary to obtain asymptotic estimates for certain exponential sums related to Kloosterman's sum. However, since these sums depend upon the multiplier system in an essential way, we must first determine all such multiplier systems corresponding to the dimension zero for each of the groups $G(\sqrt{2})$ and $G(\sqrt{3})$. This is the object of the present paper.

Specifically, let Γ denote either group and let $f(\tau)$ be a form of dimension zero for Γ. That is, $f(\tau)$ is regular in $\mathscr{I}(\tau) > 0$, and

$$(1) \qquad f(M\tau) = \epsilon(M)f(\tau), \quad \text{for all } M \text{ in } \Gamma \text{ and } \quad \mathscr{I}(\tau) > 0,$$

where ϵ does not depend on τ and $|\epsilon(M)| = 1$ for all M in Γ. We here compute all such multipliers as functions of M. Our results are contained in Theorems 1 and 2 below.

2. **Preliminaries.** Let M be an element of Γ and write

$$M\tau = (\alpha\tau + \beta)/(\gamma\tau + \delta).$$

We identify with M the matrices

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\alpha & -\beta \\ -\gamma & -\delta \end{bmatrix},$$

which are to be regarded as equal in this context because of the interpretation

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