LINEAR DIFFERENTIAL EQUATIONS AND CONVEX MAPPINGS

By Paul R. Beesack

1. **Introduction.** In a recent paper [1], R. F. Gabriel introduced a generalized Schwarzian derivative of an analytic function $f(z)$, defined as

$$\{f(z), z\}_n \equiv \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{n+1}\left(\frac{f''(z)}{f'(z)}\right)^2, \tag{1.1}$$

and used it to prove the following theorem.

Theorem I (Gabriel). *Let* $f(z) = z^{-n} + \cdots$ *be regular, with* $f'(z) \neq 0$ *in* $0 < |z| < 1$, *and let*

$$|\{f(z), z\}_n| \leq \frac{(n+1)c_n}{|z|}, \qquad 0 < |z| < 1, \tag{1.2}$$

where c_n *is an appropriate constant. Then* $f(z)$ *maps* $|z| = r < 1$ *onto a curve which is convex of order* $-n$. *If, further, there is a value* w_0 *for which* $f(z) \neq w_0$ *in* $0 < |z| < 1$, *then* $f(z)$ *is n-valent and starlike of order* $-n$ *with respect to* w_0 *in* $0 < |z| < 1$. *For each n the constant* c_n *is best possible.*

In this theorem $n > 1$. For $n = 1$ Gabriel [2] had proved a corresponding theorem using the ordinary Schwarzian derivative (to which (1.1) reduces, when $n = 1$), and with (1.2) replaced by

$$|\{f(z), z\}_1| \leq 2c_1, \qquad 0 \leq |z| < 1. \tag{1.3}$$

This theorem was generalized by D. Haimo [3]. The purpose of this paper is to prove the following theorem, which will contain both Theorem I, and the result of Haimo.

Theorem II. *Let* $f(z) = z^{-n} + \cdots$ $(n \geq 1)$ *be regular, with* $f'(z) \neq 0$ *in* $0 < |z| < 1$, *and let*

$$(n+1)p(z) \equiv \{f(z), z\}_n, \tag{1.4}$$

$$Q(r; \theta) = \Re\{e^{2i\theta}p(re^{i\theta})\}, \qquad 0 < r < 1, \qquad 0 \leq \theta < 2\pi. \tag{1.5}$$

Suppose that, for each θ, *the differential equation*

$$\frac{d^2y}{dr^2} + Q(r; \theta)y = 0, \tag{1.6}$$

has a real solution $y(r; \theta)$ *such that* $y(0; \theta) = 0$, $y(r; \theta) \neq 0$ *for* $0 < r < 1$, *and*

$$\overline{\lim_{r\to 1}}\, r\frac{y'(r; \theta)}{y(r; \theta)} \geq \frac{1}{n+1}. \tag{1.7}$$

Received February 8, 1960.