NEW CASES OF IRREDUCIBILITY FOR LEGENDRE POLYNOMIALS. II

By J. H. Wahab

Introduction. It is the purpose of this paper to extend some of the results of the author's first paper on this subject (*New cases of irreducibility for Legendre polynomials*, this Journal, vol. 19(1952), pp. 165–176.) The notation, results, and bibliography of that paper will be used. Moreover a reference to that paper, except to its bibliography, will be preceded by I. For instance, equation I-(2.7) will indicate equation (7) in §2 of that paper.

1. **A generalization of an earlier result.** In this section a natural generalization of Theorem I-3.1 will be given and will be applied to obtain a result analogous to Theorem I-3.2. $E(r)$ will be used to denote the power of the prime p dividing

$$\binom{n+r}{n}\cdot\binom{n}{r} \quad \text{and} \quad \sigma(n)$$

to denote the sum of the digits of the p-adic decimal representation of n.

Theorem 1.1. *Let* $n = \sum_{i=1}^{s} (p-1)p^{k_i}$, *$p$ a prime, integers* $k_1 > k_2 > \cdots > k_s \geq 0$; *let* $r_0 = 0$ *and* $r_i = \sum_{j=1}^{i} (p-1)p^{k_j}$ *for* $1 \leq i \leq s$; *then*

(A) $E(r_i) = i$ for $0 \leq i \leq s$, *and*

(B) $E(r) > E(r_i)$ *for* $0 \leq i < s$ *and* $r_i < r < r_{i+1}$.

Proof. By equation I-(1.7) the exponent of the power of the prime p dividing $\binom{n}{r}$ is given by $(p-1)^{-1}[\sigma(r) + \sigma(n-r) - \sigma(n)]$ or more simply by the number of borrows when r is subtracted from n in p-adic notation; equivalently, it is the number of carries when r and $n - r$ are added in p-adic notation. Let $F(r)$ be the number of times p divides $\binom{n+r}{r}$ and $G(r)$ be the number of times p divides $\binom{n}{r}$. We shall obtain $F(r)$ as the number of carries when r is added to n and $G(r)$ as the number of borrows when r is subtracted from n. Moreover $E(r) = F(r) + G(r)$. For $i > 0$, the p-adic representation of r_i will consist of $(p-1)$'s in i places and 0's elsewhere and it will terminate in at least one zero for $i < s$. Say it terminates in t 0's. As the first $k_1 + 1 - t$ places of n are the same as for r_i, $F(r_i) = i$, $G(r_i) = 0$, and (A) follows.

Likewise for $r_i < r < r_{i+1}$ the first $k_1 + 1 - t$ places of n and r are identical and $E(r) \geq i$. At least one of the last t places of r is not zero. If this non-zero digit occurs where the corresponding digit of n is $p - 1$, we get a carry in computing $n + r$. Otherwise we have to borrow to compute $n - r$. Hence $E(r) > i$.

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