PERMUTATIONS WITH COMPARABLE SETS OF INVARIANT MEANS

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1. **Introduction.** Let X denote an infinite set. A mapping $\sigma : X \to X$ will be called a *motion of* X if it is one-to-one; it need not be onto. Let E be the Banach space of all bounded real-valued functions on X, with

$$|| f || = \sup_{p \varepsilon X} | f(p) |,$$

and let E' denote the conjugate space of E. Then if $\phi' \varepsilon E'$, (ϕ', f) will denote the value of the functional ϕ' at the point $f \varepsilon E$. $L(E, E)$ will denote the algebra of bounded linear transformations of E. Associated with a motion σ is its representation in $L(E, E)$: the transformation S defined by $(Sf)(p) = f(\sigma p)$ for all $p \varepsilon X$ and all $f \varepsilon E$. Similarly, each $p \varepsilon X$ is represented in E' by the *point-functional* $p' \colon f \to f(p)$. The set of all point functionals will be denoted by X'.

Given a motion σ of X, there is defined a subset $M_\sigma' \subset E'$, the set of *σ-invariant means*; $\phi' \varepsilon M_\sigma'$ if and only if:

1a $|| \phi' || = 1$;

1b $(\phi', u) = 1$, where $u \varepsilon E$ is defined by $u(p) \equiv 1$;

1c $f \varepsilon E$, $f(p) \geq 0$ for all $p \varepsilon X \Rightarrow (\phi', f) \geq 0$;

1d $f \varepsilon E \Rightarrow (\phi', Sf - f) = 0$, where S represents σ in $L(E, E)$.

When X is the set N of positive integers, and σ is the 'translation' mapping $\sigma \colon p \Rightarrow p + 1$, an element of M_σ' is called a Banach Limit [1; 33, 34] and [7; 19ff.]. We may note in passing that any two of 1a, 1b, 1c imply the third.

The object of the present paper is to provide several answers to the following question: Given two motions σ and μ, with representations S and U, what properties of σ and μ, or of S and U, characterize the situation $M_\sigma' \subset M_\mu'$? In §2 will be given a construction of M_σ'—a modification of a construction given in [7; 21ff.]— in terms of nets of averages of S and of point-functionals. This construction is used in the proof of the main theorem, Theorem 3.3, of §3. In §4 another condition is introduced, which seems stronger than those of Theorem 3.3, though one may conjecture its equivalence. This boundedness condition is used repeatedly, after the situation is specialized to the case $X = N$ (the positive integers). It is then shown how every motion without finite orbits is equivalent to a motion having a particularly simple appearance ('single-cycle'). Here, equivalence of σ and μ means that $M_\sigma' = M_\mu'$.

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