DIVISION RINGS CONTAINING A REAL CLOSED FIELD

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A classical theorem of Frobenius asserts that the only real division algebras are the real numbers, the complex numbers, and the quaternions. More generally, it is the case that the only division algebras over a real closed field R are R itself, $R(i)$, and the quaternion algebra over R, i.e., the four-dimensional algebra over R with basis $1, i, j, ij$ and multiplication defined by $i^2 = j^2 = -1$, $ij = -ji$ (cf. Albert, [1; 145–146]). An algebra of the latter sort is commonly called an algebra of "real quaternions". The purpose of the present paper is to show that the hypothesis of the Frobenius theorem can be weakened by removing the assumption that R is in the center of the division algebra, without substantially weakening the conclusion. Specifically, it will be shown that if D is a division ring containing a real closed field R such that D is a finite dimensional left vector space over R, then either $D = R$, $D = R(i)$, or D is an algebra of real quaternions. In the latter case, however, the center of D need not be R itself but may be some other real closed field F such that $F(i)$ is isomorphic to $R(i)$. The Frobenius theorem is an immediate corollary. For if, in fact, R is in the center of D, then F contains R and is of finite dimension over R since D is, but a real closed field, from its definition, cannot be a finite algebraic extension of another. Therefore, $F = R$ and D is the quaternion algebra over R, as asserted by Frobenius. However, one does not obtain this way a new proof of the Frobenius theorem, for the proof of the present theorem is reduced to that of Frobenius' theorem by showing that under the present hypotheses, D is in fact finite-dimensional over its center and that the center is a real closed field.

1. **Real closed fields.** It is convenient to recall here certain known results on real closed fields. Most proofs can be found in van der Waerden [3, Vol. I, Chapter IX]. A real closed field R may be defined to be either a formally real field (i.e., one in which a non-trivial sum of squares can not vanish) no algebraic extension of which is formally real, or an ordered field such that $R(i)$ is algebraically closed, these things being equivalent. (Here i denotes a square root of -1.) It follows that R is of characteristic zero, and being ordered, possesses a natural topology in which the rationals can be shown to be dense and which induces on the rationals their usual topology. Therefore R in fact possesses a natural metric and is isomorphic, in an isomorphism preserving this metric, to a subfield $\mathbf{F}$ of the real numbers $\mathbf{R}$ which is algebraically closed in $\mathbf{R}$, i.e., such that any element of $\mathbf{R}$ algebraic over $\mathbf{F}$ is in $\mathbf{F}$. Conversely, any such field is real closed. Distinct real closed subfields of $\mathbf{R}$ can not be isomorphic, since were

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