DIFFERENTIABILITY AND ANALYTICITY OF FUNCTIONS IN LINEAR ALGEBRAS

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1. **Introduction.** In the literature on functions defined on linear algebras with unit element over the real or complex field, only Ward [15], Rinehart [9], and Portmann [7] have proposed definitions of derivative which do not require that the algebra be commutative. Several authors, Hausdorff [4], Ringleb [10], Wagner [14], and Volovel'skaya [13], to name only a few, employed the notion of the differential of a function to develop a function theory for linear algebras, but they were not successful in defining a derivative to go with the definition of analytic function.

In this paper a definition of derivative which was proposed by Zorn [16] in connection with functions in Banach spaces is used to develop a theory of functions in linear algebras. This approach seems to simplify and improve the works of the authors referred to above, and at the same time new light is shed on the connections among their works.

2. **Regular representations and the enveloping algebra.** We begin with a brief exposition of some of the basic ideas on the representations of algebras. Let $\mathfrak{A}$ be a linear associative algebra with unit element e over a commutative field Φ. $\mathfrak{A}$ is a finite dimensional vector space over Φ which is also a ring. Hence, for elements $x, y, z \in \mathfrak{A}$, we have

$$(x + y)z = xz + yz, \tag{2.1}$$

$$z(x + y) = zx + zy, \tag{2.2}$$

$$x(yz) = (xy)z, \tag{2.3}$$

$$(\alpha x)y = x(\alpha y) = \alpha(xy) \quad \text{for} \quad \alpha \in \Phi. \tag{2.4}$$

We associate with each element a of $\mathfrak{A}$ the mapping $x \to ax = A_l x$ of $\mathfrak{A}$ to itself. Equations (2.2) and (2.4) imply that the mapping A_l is a linear transformation on $\mathfrak{A}$, and equations (2.1), (2.3), and (2.4) imply that the correspondence $a \to A_l$ is an operator homomorphism, the so-called first regular representation of $\mathfrak{A}$ in terms of linear transformations. If $ax = 0$ for all x in $\mathfrak{A}$, then $ae = a = 0$. Hence, the representation is faithful (i.e. an isomorphism). Denote by $\mathfrak{T}_l$ the algebra of these linear transformations, the algebra of left multiplications. Then we may write $\mathfrak{A} \simeq \mathfrak{T}_l$.

The representation by matrices is obtained in the following manner. Let

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