EULERIAN NUMBERS AND POLYNOMIALS OF HIGHER ORDER

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1. Introduction. The so-called "Eulerian" numbers $H_n[\lambda]$ may be defined by means of

$$(1.1) \qquad \frac{1-\lambda}{e^x-\lambda} = \sum_{n=0}^{\infty} H_n[\lambda]\frac{x^n}{n!},$$

where the parameter $\lambda \neq 1$ but is otherwise arbitrary. This definition is equivalent to

$$(1.2) \qquad (H+1)^n = \lambda H_n \qquad (n>0), \quad H_0 = 1,$$

where after expansion of the left member, superscripts are replaced by subscripts. The polynomial $H_n(u) = H_n(u|\ \lambda)$ is defined by

$$(1.3) \qquad H_n(u) = (u+H)^n,$$

or, equivalently,

$$(1.4) \qquad \frac{1-\lambda}{e^x-\lambda}e^{xu} = \sum_{n=0}^{\infty} H_n(u \mid \lambda)\frac{x^n}{n!}.$$

Nörlund [7, Chapter 6] has defined Bernoulli and Euler polynomials of order k. They may be defined by means of the following generating relations:

$$(1.5) \qquad \frac{\omega_1\omega_2\cdots\omega_k x^k e^{xu}}{(e^{\omega_1 x}-1)\cdots(e^{\omega_k x}-1)} = \sum_{n=0}^{\infty} B_n^{(k)}(u)\frac{x^n}{n!},$$

$$(1.6) \qquad \frac{2^k e^{xu}}{(e^{\omega_1 x}+1)\cdots(e^{\omega_k x}+1)} = \sum_{n=0}^{\infty} E_n^{(k)}(u)\frac{x^n}{n!},$$

where

$$B_n^{(k)}(u) = B_n^{(k)}(u \mid \omega_1, \cdots, \omega_k), \qquad E_n^{(k)}(u) = E_n^{(k)}(u \mid \omega_1, \cdots, \omega_k).$$

In particular the Bernoulli and Euler numbers of order k are given by

$$B_n^{(k)} = B_n^{(k)}[\omega_1, \cdots, \omega_k] = B_n^{(k)}(0 \mid \omega_1, \cdots, \omega_k)$$

$$E_n^{(k)} = E_n^{(k)}[\omega_1, \cdots, \omega_k] = 2^n E_n^{(k)}\, \frac{\omega_1+\cdots+\omega_k}{2} \,\Big|\, \omega_1, \cdots, \omega_k$$

It is natural to define $H_n^{(k)}(u)$ by means of

$$(1.7) \qquad \frac{(1-\lambda_1)\cdots(1-\lambda_k)\,e^{xu}}{(e^{\omega_1 x}-\lambda_1)\cdots(e^{\omega_k x}-\lambda_k)} = \sum_{n=0}^{\infty} H_n^{(k)}\ (u)\frac{x^n}{n!},$$

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