SEPARABILITY IN METRIC SPACES

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In his dissertation (The University of Texas, 1958) the author proved the following theorem: If Σ is a connected metric space which is (1) locally peripherally separable [1], and (2) compactly connected [3], then Σ is completely separable. While considering the effect of replacing conditions (1) and (2) of the hypothesis by a single condition, Moore's axiom 5 in "Foundations," the author discovered the following theorem: If Σ is a connected metric space such that (1) no point separates space, and (2) if P and Q are two points and R is a region containing P, then there exists in R a compact continuum M which separates P from Q, then Σ is completely separable.

Definition. For each positive integer n, let G_n denote the collection of all open sets having diameter less than $1/n$.

Lemma 1. *If P is a point of the region R and M is a closed and compact point set not containing P, then there exists a compact continuum N lying in R such that N separates P from M.*

Proof. Let Q denote a point of M. In G_1 there is a subset R_1 of R which (i) contains P and (ii) contains a compact continuum N_1 which separates P from Q. $S - N_1 = N(P, 1) + N(Q, 1)$, where $N(P, 1)$ and $N(Q, 1)$ are disjoint open sets containing P and Q, respectively. In G_2 there is a subset R_2 of $N(P, 1) \cdot R_1$ which (i) contains P and (ii) contains a compact continuum N_2 which separates P from $N(Q, 1) + N_1$. $S - N_2 = N(P, 2) + N(Q, 2)$, where $N(P, 2)$ and $N(Q, 2)$ are disjoint open sets containing P and Q, respectively. In G_3 there is a subset R_3 of $N(P, 2) \cdot R_2$ which (i) contains P and (ii) contains a compact continuum N_3 which separates P from $N(Q, 2) + N_2$. Consider a continuation of this process.

Let $N_P = N(P, 1) \cdot N(P, 2) \cdot \cdots$ and $N_Q = \Sigma N(Q, i)$. Suppose N_P is non-degenerate. Since space is connected, and no point of N_Q is a limit point of N_P, some point T of $N_P - P$ is a limit point of N_Q, or else P would separate space. Since $P + \Sigma N_i$ is closed, there exists a region R containing T, but no point of this set. There exists in R a compact continuum L separating T from P. $S - L = L_P + L_T$, where L_P and L_T are disjoint open sets containing P and T, respectively. Since P is a limit point of ΣN_i, there is a positive integer d such that N_d is a subset of L_P. There exists a positive integer $j > d$ such that L_T contains a point of $N(Q, j)$, since L_T is open. But $L + L_T$ is a connected subset of $S - N_j$ which intersects $N(Q, j)$. Therefore, $L + L_T$ is a subset of

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