DIRICHLET ALGEBRAS

By John Wermer

Introduction. Let X be a compact Hausdorff space and $C(X)$ the Banach algebra of all continuous complex-valued functions on X. Let A be a closed subalgebra of $C(X)$ and assume

(1) 1 is in A

(2) A separates points on X.

The general theory of normed rings, together with our assumptions (1) and (2), now gives us the following:

Let M denote the space of all multiplicative linear functionals on A, taken in Gelfand's Topology. Then M is a compact Hausdorff space and X is homeomorphically embedded in M as a closed subset. Every f in A admits a continuous extension from X to M, by setting $f(m) = m(f)$ for all m in M. Further $| f(m)| \leq \max_{t \varepsilon X} | f(t)|$ if $m \varepsilon M, f \varepsilon A$.

Thus the functions f of the algebra A all satisfy a *maximum principle* on M, relative to the subset X. This suggests that the functions of A are *analytic* on $M - X$ in some sense. For instance, if X is the unit circle $| z | = 1$ and A the algebra on X generated by the function z, then $M - X$ can be identified with the open disk $| z | < 1$, and all functions of A are analytic there in the usual sense. A somewhat more complicated example is the following:

Let $R \times C$ be the space of pairs (t, z) with t real and z complex. Let X be the subset of $R \times C$ defined by $0 \leq t \leq 1$, $| z | = 1$. Let A be the algebra on X generated by the coordinate functions t and z. Then M can be identified with the solid cylinder: $0 \leq t \leq 1$, $| z | \leq 1$. Each f in A, when restricted to a disk: $t =$ constant, $| z | < 1$, is analytic in z. Thus the analyticity we are seeking must here be taken with respect to suitable subsets of $M - X$.

How are we to show, for an arbitrary algebra A having $M - X$ non-empty, that there exist subsets of $M - X$ on which each f in A is analytic? It is at the present time an open and apparently difficult question whether this can always be done. Gleason in [5] has suggested a promising approach to this question. He makes the following definitions:

Definition 1. Let $x, y \varepsilon M$. Then $x \sim y$ if there is a constant $k < 2$ such that $| f(x) - f(y)| \leq k$, all $f \varepsilon A$ with $\max_X | f | \leq 1$.

Gleason shows that the relation $\sim$ is an equivalence relation in M.

Definition 2. *A part* of M is an equivalence class of M under the relation $\sim$.

In many concrete examples, certain of these parts turn out to be the desired subsets of $M - X$ on which the functions of A are analytic. For instance, in the above example with M the cylinder: $0 \leq t \leq 1$, $| z | \leq 1$, each disk: $t =$ constant, $| z | < 1$, is a part of M.

Received November 13, 1959. This work was supported by N. S. F. grant G–5866.