THE WIENER-HOPF EQUATION WHOSE KERNEL IS A PROBABILITY DENSITY. II

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1. Introduction. This paper is a continuation of an earlier one with the same title [2]. The sharpest results in [2] concern the integral equation

$$F(x) = \int_0^\infty k(x-y)F(y)\,dy, \qquad x > 0, \tag{1.1}$$

where the known kernel function $k(x)$ is assumed to be an even probability density, i.e.

$$k(x) = k(-x) \geq 0, \int_{-\infty}^{\infty} k(x)\,dx = 1. \tag{1.2}$$

The results of Theorems 5 and 6 of [2] may be summarized as follows. The integral equation (1.1) has one unique non-decreasing solution, i.e. all other monotone solutions are multiples of each other. Only the trivial solution $F(x) = 0$ for $x > 0$ has $F(0^+) = 0$ so that we may eliminate all ambiguity by denoting by $F(x)$ that non-decreasing solution of (1.1) with

$$F(0^+) = \lim_{x \to 0^+} F(x) = 1.$$

Then the Laplace transform of $F(x)$ is given by

$$\chi(\lambda) = 1 + \int_{0+}^\infty e^{-\lambda x}\,dF(x) = \exp\left\{-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\lambda}{\lambda^2+\xi^2}\log\,[1-\varphi(\xi)]\,d\xi\right\}, \tag{1.3}$$

where

$$\varphi(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}k(x)\,dx, \qquad \xi \text{ real}.$$

It should be remarked that the integral on the right in (1.3) exists as

$$\frac{\lambda}{\lambda^2+\xi^2}\log\,[1-\varphi(\xi)] \in L_1(-\infty, \infty)$$

for every $\lambda > 0$, and that $F(x)$ is absolutely continuous with respect to Lebesgue measure. For large x the behavior of $F(x)$ is described by

$$\lim_{x\to\infty} \frac{F(x)}{x} = \frac{\sqrt{2}}{\sigma}, \tag{1.4}$$

where

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