SOME IRREDUCIBILITY THEOREMS FOR BERNOULLI POLYNOMIALS OF HIGHER ORDER

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1. Introduction. The Bernoulli polynomials of order k, where k is a positive integer, are defined by

$$\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{m=0}^{\infty} B_m^{(k)}(x) \frac{t^m}{m!}.$$

The constant terms of these polynomials, $B_m^{(k)} = B_m^{(k)}(0)$, are called the Bernoulli numbers of order k. There is very little known concerning the factorization properties of the Bernoulli polynomials over the field of rational numbers (throughout this paper all references to irreducibility and factorization will be with respect to the rational field). Some obvious factors occur, and it has been conjectured that aside from these factors, the Bernoulli polynomials are irreducible. Carlitz [1] has obtained some irreducibility theorems when $k = 1$, and in this paper we obtain some like results for the polynomials of higher order. The proofs given by Carlitz in [1] depend greatly upon the fact that the Stuadt-Clausen theorem [3; 33] holds for the Bernoulli numbers of order one. When $k > 1$, nothing so definite as the Staudt-Clausen theorem is known, but we are able to make use of some properties of the Bernoulli numbers of higher order obtained by Carlitz in [2] to obtain our results.

We shall make almost constant use of the Eisenstein criterion: if

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

is a polynomial with rational coefficients, and for some prime p all coefficients are integers (mod p), $p \mid a_i$ for $i = 0, 1, \cdots, n-1$, $p^2 \nmid a_0$, and $p \nmid a_n$, then $f(x)$ is irreducible. A polynomial satisfying these conditions will be called an Eisenstein polynomial. At one point we shall use the following criterion of Perron [5, vol. 1; 195]: if for some prime p all coefficients of $f(x)$ are integers (mod p), $p \mid a_i$ for $i = 0, 1, \cdots, s-1$, $p^2 \nmid a_0$, and $p \nmid a_s$, then $f(x)$ has an irreducible factor of degree $\geq s$.

We shall also make repeated use of the fact that if $m = a_0 + a_1p + \cdots$ $(0 \leq a_i < p)$, $r = b_0 + b_1p + \cdots$ $(0 \leq b_i < p)$, then the binomial coefficient $\binom{m}{r}$ is prime to p if and only if $a_i \geq b_i$ for $i = 0, 1, 2, \cdots$.

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