STOCHASTIC MATRICES WITH A NON-TRIVIAL GREATEST POSITIVE ROOT

By Alfred Brauer

A square matrix $A = (a_{\kappa\lambda})$ of order n with non-negative elements is called stochastic if all the row-sums equal 1. It is well known that A has the trivial characteristic root 1. If A is unreduced (see [1]), then 1 is a simple root. Suppose that a_{ii} and a_{jj} are the smallest main diagonal elements of A. Then all characteristic roots of A lie in the interior or on the boundary of the oval of Cassini

$$| z - a_{ii} | \, | z - a_{jj} | \leq (1 - a_{ii})(1 - a_{jj})$$

[2, Theorems 24, 21 and 23].

Suppose that A is unreduced. Let $h_1 , h_2 , \cdots , h_n$ be arbitrary numbers. Denote the matrix $(a_{\kappa\lambda} - h_\lambda)$ by B. Then B has the trivial characteristic root

$$\omega' = 1 - \sum_{\lambda=1}^{n} h_\lambda .$$

The characteristic roots of A coincide with those of B except that the simple root 1 of A is replaced by ω' in B [2, Theorem 29].

If M is a matrix with non-negative elements whose row-sums all equal s, then M is called a generalized stochastic matrix. Since we can write

$$M = sA$$

where A is a stochastic matrix, the characteristic roots of M are obtained by multiplying those of A by s [2, Theorem 32].

In this paper, I consider classes of unreduced stochastic and generalized stochastic matrices which have a non-trivial greatest positive root η which is smaller than the trivial root, but greater than or equal to the absolute value of all the other roots. Since η is obtained as greatest positive root of another positive or non-negative matrix, the theorems of Frobenius [6, 7], Ledermann [8], Ostrowski [9], and myself [3] can be used to obtain estimates for η. Moreover, the method which I obtained in my paper [5] can be used to compute η with or without computing machines as exactly as needed.

Theorem 1. *Suppose that the unreduced generalized stochastic matrix $A = (a_{\kappa\lambda})$ has the property that the off-diagonal elements of one row are less than or equal to the other elements in the same column, then A has a positive non-trivial characteristic root η which is greater than or equal to the absolute value of all the other non-trivial characteristic roots of A unless all non-trivial roots of A vanish.*

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