THE EQUATION $\bar{t}at = b$ IN A QUATERNION ALGEBRA

By Barth Pollak

Introduction. The quaternion equation $\bar{t}at = b$ has been studied by O'Connor and Pall [5], [7] for the case of classical (Hamilton) quaternion algebras over rational p-adic fields. Here a and b are non-zero quaternions having zero trace and non-zero norm. They obtained necessary and sufficient conditions for solvability of the equation. They also found that if $p > 2$, then the equation may not be solvable for some quaternion t. However, if it is solvable, then there exist solutions t with Nt assuming either of the two values permitted by the norm condition, $Nb = (Nt)^2 Na$. However, for $p = 2$ it was found that the equation was always solvable (provided, of course, that a and b satisfy the above norm condition). However, Nt in this case was invariant. Since the classical quaternion algebra splits over odd p-adic fields and is a division algebra over the 2-adic field, Pall has made the following natural conjecture. Let Q be a quaternion algebra over the rationals and Q_p its scalar extension over the rational p-adic field. If Q_p splits and the equation $\bar{t}at = b$ is solvable, Nt assumes both values permitted by the norm condition. If Q_p does not split, the equation is always solvable but in this case Nt is invariant. In our investigation we give necessary and sufficient conditions for the solvability of $\bar{t}at = b$ for any quaternion algebra over an arbitrary ground field of characteristic $\neq 2$. We also derive a result which, when k is specialized to a local field, gives Pall's conjecture. Finally, we treat analogous questions for a maximal order M within a quaternion algebra.

1. **Notations.** In this paper k will denote a field of characteristic $\neq 2$, k^* the multiplicative group of non-zero elements of k. Elements of k will be denoted by small case Greek letters. Q will denote a quaternion algebra over k. Thus Q is a 4-dimensional associative algebra over k with basis $1, i_1, i_2, i_3 = i_1 i_2$, and the multiplication table is $i_1^2 = \alpha$, $i_2^2 = \beta$, $i_1 i_2 = -i_2 i_1$. For uniformity of notation we will sometimes denote 1 by i_0. It will be convenient to adopt the notation (α, β) for Q. Elements of Q will be denoted by small case Latin letters. There is an anti-automorphism of period two called *conjugation* in Q. Thus, if $x = \xi_0 i_0 + \xi_1 i_1 + \xi_2 i_2 + \xi_3 i_3$, then $\bar{x}$, the *conjugate* of x, is given by $\bar{x} = \xi_0 i_0 - \xi_1 i_1 - \xi_2 i_2 - \xi_3 i_3$. The quantities $x + \bar{x}$ and $x\bar{x}$ are scalar multiples of i_0, and in this sense we say they lie in k (by making the natural identification). They are called, respectively, the *trace* and *norm* of x and are denoted by Sx and Nx. If $Sx = 0$, x is called *pure*.

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