CONVOLUTIONS AND GENERAL TRANSFORMS ON L^p

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1. Introduction. It has been shown by Titchmarsh [6;93] that if f and g are a pair of Fourier cosine transforms in $L^2(0, \infty)$, then so are

$$F(x) = \int_x^\infty \frac{f(y)}{y}\, dy \quad \text{and} \quad G(y) = \frac{1}{y}\int_0^y g(x)\, dx.$$

In a recent paper [3] the author has generalized this result showing that if f and g are a pair of cosine transforms in L^2, then so are

$$F(x) = \int_0^\infty \frac{1}{y}\varphi\left(\frac{x}{y}\right) f(y)\, dy \quad \text{and} \quad G(y) = \frac{1}{y}\int_0^\infty \varphi\left(\frac{x}{y}\right) g(x)\, dx.$$

Here φ is any non-negative function satisfying $\int_0^\infty \varphi(y) y^{-\frac{1}{2}}\, dy < \infty$. (The result of Titchmarsh is the special case $\varphi(x) = 1$ $(0 \leq x \leq 1)$, $\varphi(x) = 0$ $(1 < x < \infty)$.)

One object of the present paper is to show that the generalized result applies not only to the cosine transform but to *any* Watson transform (and indeed to a more general class of transforms).

We shall make use of a theorem of Schur [5] concerning integral transforms with kernels homogeneous of degree -1 (see Section 3). The first part of our paper is devoted to a demonstration of a theorem (Theorem 2.8) concerning functions on n-dimensional space which reduces to the theorem of Schur when $n = 1$. This n-dimensional theorem is obtained by first constructing an isomorphism between L^p spaces on all of n-space and certain spaces of functions defined on the set of n-tuples with positive coordinates and then applying a well-known result concerning L^p spaces on arbitrary locally compact Abelian groups. Thus we put Schur's classical result in a group theoretic setting.

Finally, making use of the isomorphism mentioned above, together with our result on Watson transforms, we easily deduce the theorem of Bochner and Chandrasekharan [1;170] which states that every Watson transform is the product of a certain "elementary" Watson transform and a unitary operator which (when considered as acting on $L^2(-\infty, \infty)$) commutes with translations.

2. Preliminaries on some functions spaces. We shall write (x_1, x_n) for $(x_1, x_2, \cdots, x_n)$. Similarly $x_1x_2 \cdots x_n$, $dx_1 \cdot dx_2 \cdots dx_n$, and $x_1 + x_2 + \cdots + x_n$ will be abbreviated as x_1x_n, dx_1dx_n, $x_1 + x_n$ respectively.

Now for $n = 1, 2, \cdots$, let E^n denote Euclidean n-space. That is, E^n is the set of all n-tuples of real numbers (x_1, x_n). If we define addition of n-tuples as addition coordinate-wise, then E^n (with the usual metric) becomes a locally compact group for which Haar measure is Lebesgue measure $dx_1\, dx_n$.

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