SOME PROPERTIES OF THE FUNDAMENTAL SOLUTION OF THE PARABOLIC EQUATION

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1. **Introduction.** The existence of a fundamental solution of the parabolic equation

$$\begin{aligned} L(u) \equiv & \sum_{i,j=1}^{n} a_{ij}(x_1, \cdots, x_n, t)\, \partial^2 u/\partial x_i\, \partial x_j \\ & + \sum_{i=1}^{n} a_i(x_1, \cdots, x_n, t)\, \partial u/\partial x_i + a(x_1, \cdots, x_n, t)u - \partial u/\partial t = 0 \end{aligned} \tag{1}$$

has been demonstrated by many authors under various continuity conditions on the coefficients. Dressel [4], [5] proved that a fundamental solution of (1) exists in a domain of n-space if the functions $\partial^2 a_{ks}/\partial x_i \partial x_j$ and $\partial a_{ij}/\partial_t$ are bounded and satisfy Hölder conditions with respect to all $n + 1$ variables. Yosida [16] and Itô [10], [11] considered the problem of finding fundamental solutions on differential manifolds. When the coefficients are infinitely differentiable, Yosida has shown that (1) has a fundamental solution in a compact subset of a Riemannian space. Ito constructs a fundamental solution on a differentiable manifold of class C^2 under the same continuity conditions (though with somewhat different boundedness conditions) as were required in the papers by Dressel. Pogorzelski [13] has shown that a fundamental solution of (1) can be found in a bounded domain in n-space even though the a_{ij} may not be differentiable. His main requirements on the a_{ij} are that they satisfy a Hölder condition in all the variables $x_1, \cdots, x_n$, and t. In a later paper [14] he extends his solution to all of n-space in the special case that $a_{ij} = \delta_{ij}$ outside of a fixed bounded region. In a recent paper, Aronson [1] proves the existence of a fundamental solution in all of n-space in case the functions a_{ij}, a_i, a, $\partial a_i/\partial x_r$ are continuous in t and satisfy uniform Hölder conditions with respect to the space variables. In an appendix to his paper, Aronson states that the restrictions that a_i have Hölder continuous first derivatives can be removed, and then outlines the changes needed to accomplish this. In the present paper the existence of a fundamental solution is proved for both bounded and unbounded domains under conditions similar to those imposed by Aronson. Our methods are different, and, since we start with a different parametrix, the resulting fundamental solutions are not the same. Making use of several lemmas in the existence proof of the fundamental solution Γ, we next derive several of the

Received July 20, 1959; presented to the American Mathematical Society, November 27, 1958. This research was supported by the United States Air Force through the Air Force Office of Scientific Research and Development Command, under contract No. AF 18(600)-1341.