SOME CANCELLATION THEOREMS FOR ORDINAL PRODUCTS OF RELATIONS

By Chen Chung Chang and Anne C. Morel

In this paper we are concerned with cancellation in ordinal products of reflexive relation types. Consider the following statements: (i) If $\alpha \cdot \beta = \alpha \cdot \gamma$, then $\beta = \gamma$, and (ii) If $\beta \cdot \alpha = \gamma \cdot \alpha$, then $\beta = \gamma$. It will be shown that (i) holds when α is a finite non-zero type and that (ii) holds when all three types are finite and $\alpha \neq 0$. We obtain several refinements of these two results and also give examples of types for which either (i) or (ii) fails.

As an introduction we present very briefly a collection of some basic definitions and results. (It should be pointed out that most of the notions we are about to introduce are fairly well known in the mathematical literature, see [1], [4] and [5]; for a reference where most of these can be found in a single body, we refer the reader to [4]. Our notation in the main agrees with that of [4].) By a *binary relation* (or simply a *relation*) R, we mean a set of ordered couples $\langle x, y \rangle$. The letters R, S, T, U, $\cdots$ will be used as variables to range over relations; the symbol 0 shall denote both the empty set and the empty relation. By the *field* $F(R)$ *of a relation* R, we mean the set of all elements x where either $\langle x, y \rangle \in R$ or $\langle y, x \rangle \in R$ for some y. We sometimes use the notation xRy to stand for the expression $\langle x, y \rangle \in R$ and the notation $x\bar{R}y$ to stand for the expression $\langle x, y \rangle \notin R$. The *Cartesian product* $M \times N$ *of sets* M *and* N is the set of all ordered couples $\langle x, y \rangle$ where $x \in M$ and $y \in N$. A relation R is *reflexive* in case xRx for each $x \in F(R)$. We adopt the convention, for this paper, that *all relations henceforth are assumed to be reflexive.* Clearly, a relation R is finite, or infinite, if and only if $F(R)$ is finite, or infinite. R is a *cardinal relation*, in symbols $R \in CR$, in case xRy implies $x = y$. R is a *square relation*, in symbols $R \in SR$, in case $R = F(R) \times F(R)$. We let OR denote the class of all simply ordering relations. If a set M is included in the domain of a function f, we let $f^*(M)$ denote the set of elements $f(x)$ for $x \in M$. Two relations R and S are *isomorphic*, in symbols $R \cong S$, if there exists a one-to-one mapping f of $F(R)$ onto $F(S)$ (whence $f^*(F(R)) = F(S)$) such that for each $x, y \in F(R)$, xRy if and only if $f(x)Sf(y)$. It is evident that $\cong$ is an equivalence relation; furthermore, if we let $\tau(R)$ denote the equivalence class determined by R under $\cong$, we see that $\tau(R) = \tau(S)$ if and only if $R \cong S$. $\tau(R)$ is called the *type of the relation* R. Quite obviously, any theorem stated in terms of isomorphism of relations can also be stated in terms of equality of the corresponding types, and vice-versa. Types of relations shall be denoted by the small Greek letters α, β, γ, δ, $\cdots$. A type $\alpha = \tau(R)$ is finite, or infinite, in case R is finite, or infinite. We define

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