STATIONARY POINTS FOR FINITE TRANSFORMATION GROUPS

By John Greever

1. Introduction.

(1.1) Definition. If T is a transformation of the space X into itself, then F_T is the set of points in X which are fixed under T. If G is a group of transformations of the space X into itself, then F_G is the set of points in X which are fixed under every element of G.

R, Z and Z_p respectively are the additive groups of rationals, integers and integers modulo the prime p. All homology groups are to be understood in the sense of Čech, the 0-dimensional groups being reduced, and all transformations are to be taken as continuous.

Let X be a finite-dimensional compact Hausdorff space and let G be a group of transformations of X into itself. A corollary to the Lefschetz Fixed Point Theorem asserts that F_G is non-empty when G is cyclic and X is triangulable and homologically trivial over R. From a theorem of P. A. Smith it follows that F_G is non-empty when X is homologically trivial over Z_p and G is of order p^α. It is the purpose of this paper to set forth results of this nature for groups of more arbitrary orders than those dealt with by the Lefschetz and Smith Theorems.

2. Normal chains with prime power factors. In discussing the existence of a stationary point for a finite group G of transformations, one is led to consider the existence of stationary points for normal subgroups of G. The definition given below serves to facilitate the statement of certain theorems; the lemmata are introduced at this point to help clarify the definition and will be used to advantage in the sequel.

(2.1) Definition. A normal chain

$$G_0 = G \supset G_1 \supset G_2 \supset \cdots \supset G_n = \{e\}$$

for the group G is of *type* $(p_1, p_2, \cdots, p_k)$, where p_i is prime for each i, if and only if there exist integers $n_0 = 0 < n_1 < n_2 < \cdots < n_k$ such that the order of G_j/G_{j+1} is a non-negative power of p_i whenever $n_{i-1} \leq j < n_i$. Furthermore, the normal chain is *proper relative to* p_i if and only if n_{i-1} and n_i are consecutive integers and the order of $G_{n_{i-1}}/G_{n_i}$ is p_i.

(2.2) Lemma. *Let G be a group of order $p^\alpha q$, where p and q are primes and α a non-negative integer. Then G has a normal chain of type (p, q, p) which is proper relative to q.*

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