DIRECT SUMS OF COUNTABLE GROUPS

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1. **Introduction.** Let G be a reduced primary Abelian group. Suppose that G is countable; then Ulm's theorem [5] asserts that G is determined up to isomorphism by its Ulm invariants. If, instead of assuming that G is countable, we assume that G is a direct sum of (any number of) cyclic groups, i.e., a direct sum of finite groups, then G is again determined by its Ulm invariants. The main purpose of this paper is to unite these two cases by proving the following generalization of Ulm's theorem: *If G is a direct sum of countable groups, then G is determined by its Ulm invariants.*

In §3 we consider a companion question, that of the existence of a group with prescribed Ulm invariants. Necessary and sufficient conditions are given for the existence of a reduced primary Abelian group which is a direct sum of countable groups and has the prescribed invariants as its Ulm invariants.

The problem of determining when a given primary Abelian group is a direct sum of countable groups is of interest in view of the isomorphism theorem mentioned. In §5 we prove a theorem along these lines for one special case.

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2. **Basic notions.** We recall that an Abelian group G is *primary* if for a fixed prime p, the order of each element in G is a power of p. To study primary Abelian groups, it usually suffices to study those which have no (non-zero) divisible subgroups, i.e., no subgroups $S \neq 0$ with $pS = S$. In this case we call G *reduced.*

For every primary Abelian group G we have a descending chain of subgroups G_α , one for each ordinal number α, defined as follows:

$G_0 = G$

$G_\alpha = pG_\beta$ if $\alpha = \beta + 1$

$G_\alpha = \bigcap_{\beta < \alpha} G_\beta$ if α is a limit ordinal.

Let λ be the first ordinal for which $G_\lambda = G_{\lambda+1}$. Then G_λ is divisible. If G is reduced, G_λ must be 0 and λ is called the *length* of G.

An element x in G has *infinite height* if x is in $G_\omega = \bigcap_{n<\omega} G_n$. (We shall not need the more refined notion of height given in [1; 28].) If x is not in G_ω , the *height* of x is n if x is in $G_n = p^nG$ but not in $G_{n+1} = p^{n+1}G$. In this case we write $h(x) = n$. A subgroup H of G is *pure* if $p^nG \cap H = p^nH$ for all n. If H is pure, an element of H has the same height in H that it has in G.

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