A GENERALIZATION OF A FUNCTIONAL EQUATION RELATED TO THE THEORY OF PARTITIONS

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1. Introduction. In a recent issue of this journal the author [1] proved the following functional equation:

$$
\begin{aligned}
(1.1)\qquad &\sum_{l=0}^{\infty} \{\lambda((l+\alpha)z - i\beta) + \lambda((l+1-\alpha)z + i\beta)\} + \pi z(\alpha^2 - \alpha + \tfrac{1}{6}) \\
&\qquad = \sum_{l=0}^{\infty} \{\lambda((l+\beta)/z + i\alpha) + \lambda((l+1-\beta)/z - i\alpha)\} \\
&\qquad\quad + (\pi/z)(\beta^2 - \beta + \tfrac{1}{6}) + 2\pi i(\alpha - \tfrac{1}{2})(\beta - \tfrac{1}{2}),
\end{aligned}
$$

where $0 \leq \alpha \leq 1, 0 < \beta < 1$ (or $0 < \alpha < 1, 0 \leq \beta \leq 1$), z is a complex number with $\Re(z) > 0$, and $\lambda(t)$ denotes $-\log(1 - e^{-2\pi t})$, the logarithm having its principal value.

Equation (1.1) may be used to obtain a simple proof of the transformation formula for the Dedekind modular function (see [1]). This transformation formula, on the other hand, was generalized in 1934 by E. M. Wright [4; 149, Theorem 4] in connection with the analytic theory of partitions (cf. Schoenfeld [2]).

The main purpose of the present paper is to develop a generalization of (1.1) in the direction of Wright's formula above. The resulting new functional equation appears in a theorem (§2). A proof of this theorem will be given in §4. In §5 we shall derive Wright's formula as an easy consequence of our theorem, and thus we have a comparatively simplified proof of Wright's formula.

It may be mentioned, in passing, that another application of the equation (1.1) has been found in the derivation of a transformation formula related to a certain type of partition function (see [5]). The generalized functional equation to be obtained in this paper would also have an analogous application to the theory of partitions; this remains to be investigated.

Throughout the paper we will use the following notation: $\kappa =$ a positive integer;

$$
\epsilon_{\kappa,s} = -ie^{\pi i(s-\frac{1}{2})/\kappa}, \qquad \beta_s = \begin{cases} \beta & (s \text{ odd}), \\ 1-\beta & (s \text{ even}), \end{cases} \qquad (s = 1, 2, \cdots, \kappa);
$$

$\lambda(t) = -\log(1 - e^{-2\pi t})$ (the principal value),
$B_\nu(t) =$ the ν-th Bernoulli polynomial,
$B_\nu =$ the ν-th Bernoulli number,
$\Gamma(t) =$ the gamma-function.

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