PAIRS OF COMMUTING MATRICES OVER A FINITE FIELD

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In this paper, we determine the number of ordered pairs of commuting n by n matrices over $GF(q)$ and give a simple generating function for this number.

To state the main result we need some notation. For any partition π of n, let $b_i \geq 0$ denote the frequency of the part $i \geq 1$, so that $n = b_1 + 2b_2 + \cdots$, and in the usual notation, $\pi = (1^{b_1}2^{b_2} \cdots)$. Let $k(\pi)$ be the total number of parts in π, that is,

$$k(\pi) = \sum_{i \geq 1} b_i .$$

Let

$$f(n, q) = f(n) = \left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{q^2}\right) \cdots \left(1 - \frac{1}{q^n}\right) \qquad (n \geq 1),$$

$$f(0) = 1.$$

Then we have the following theorem:

Theorem. *If $P(n, q) = P(n)$ is the number of ordered pairs of (not necessarily distinct) commuting n by n matrices with coefficients in $GF(q)$, then*

$$P(n) = q^{n^2} f(n) \sum_{\pi(n)} \frac{q^{k(\pi)}}{f(b_1)f(b_2) \cdots f(b_n)}. \tag{1}$$

Also,

$$\sum_{n \geq 0} \frac{P(n)}{q^{n^2} f(n)} x^n = \prod_{\substack{i \geq 1 \\ j \geq 0}} \frac{1}{(1 - q^{1-j}x^i)}. \tag{2}$$

Proof. If A is a linear transformation acting on an n-dimensional vector space V (over any field), then (see, for instance [5; 9]) $V = K_A \oplus I_A$, where

$$K_A = \{v \in V \mid A^n v = 0\},$$

$$I_A = \{v \in V \mid v = A^n w \quad \text{for some} \quad w\}.$$

The following is an immediate consequence of this decomposition.

Lemma 1. *A linear transformation B commutes with A if and only if*

(i) $B(K_A) \subset K_A$, $B(I_A) \subset I_A$

and

(ii) $B \mid K_A$ *commutes with* $A \mid K_A$, $B \mid I_A$ *commutes with* $A \mid I_A$.

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