SOME CONGRUENCES INVOLVING SUMS OF BINOMIAL COEFFICIENTS

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1. Adem [1; 233–237] has proved the following congruence:

$$\sum_{k=0}^{n} \binom{a - k(q-1)}{k} \binom{b + k(q-1)}{n-k} \equiv \binom{a+b+1}{n} \pmod{q}. \tag{1}$$

Here q is prime, $n \geq 0$, and a, b are arbitrary integers. Since

$$\binom{a - k(q-1)}{k} = (-1)^n \binom{kq - a - 1}{k},$$

(1) is easily seen to be equivalent to

$$\sum_{k=0}^{n} \binom{a + kq}{k} \binom{b + (n-k)q}{n-k} \equiv \binom{a+b+nq}{n} \pmod{q}. \tag{2}$$

We should like to point out that the congruence (2) is an immediate consequence of certain known algebraic identities. Indeed, in this way we find that (2) holds for arbitrary integers q (not necessarily prime) and also for a, b rational numbers whose denominators are prime to q.

Put

$$A_k(a, q) = \frac{a}{a + kq} \binom{a + kq}{k}.$$

Then we have the identity

$$\sum_{k=0}^{n} A_k(a, q) A_{n-k}(b, q) = A_k(a + b, q).$$

Moreover

$$\sum_{k=0}^{\infty} A_k(a, q) = x^a, \tag{3}$$

$$\sum_{k=0}^{\infty} \binom{a + kq}{k} z^k = \frac{x^{a+1}}{(1-q)x + q}, \tag{4}$$

where

$$x - 1 = x^q z. \tag{5}$$

For proof of these formulas as well as a number of applications and a detailed

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