FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING—X: RESOLVENTS, CHARACTERISTIC FUNCTIONS AND VALUES

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1. **Introduction.** In previous papers, we have shown how the functional equation technique of dynamic programming can be applied to derive variational relations for kernels and Green's functions. In [1], the Green's function associated with the second order equation

$$u'' + q(x)u = 0, \qquad u(a) = u(1) = 0, \tag{1}$$

was discussed, while in [4], analogous methods were applied to partial differential operators to obtain the Hadamard variational formula. In [2], the Fredholm integral equation was treated by similar means, and Jacobi matrices were discussed in [3].

In this paper, we wish to present some extensions of these results. Introducing the parameter λ, we consider the general equations

$$(p(x)u')' + (r(x) + \lambda q(x))u = 0, \qquad u(a) = 0, \qquad u(1) = 0, \tag{2}$$

obtaining, as in the papers cited above, variational equations for the resolvent operator as a function of a. Utilizing the meromorphic nature of the operator as a function of λ, we are able in this way to derive variational equations for the characteristic values and functions.

Corresponding results are obtained for the vector-matrix system

$$(P(t)x')' + \lambda Q(t)x = 0. \tag{3}$$

In both cases, certain assumptions have to be made concerning the coefficient functions, $q(x)$ and $Q(t)$, in order to be able to consider the differential equations as the Euler equations of associated variational problems. Since, however, we know, from the results of Miller and Schiffer, [6], concerning Green's functions for general linear differential operators of order n, (where quite different methods are employed) and from corresponding results for Fredholm operators and Jacobi matrices, that the relations obtained hold under far weaker assumptions, the interesting problem arises of deriving the more general results by variational techniques. In this paper, we present a method of analytic continuation which reduces the case of continuous $q(x)$ to that of negative continuous $q(x)$, and the case of continuous symmetric $Q(t)$ to negative definite $Q(t)$.

In a separate paper, [5], we have sketched extensions of this technique of analytic continuation which enable us to treat non-selfadjoint differential operators and non-symmetric matrices by means of variational methods.

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