ON NILALGEBRAS AND LINEAR VARIETIES OF NILPOTENT MATRICES. II

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Let F be a field and $\mathfrak{A}$ be an algebra over F, not necessarily finite dimensional and not necessarily associative. The powers of an element a in $\mathfrak{A}$ are in general not well-defined, but we shall say that $\mathfrak{A}$ is a nilalgebra if given a, there exists an integer t such that every product of t factors each of them equal to a, in whatever association, vanishes. If t can be chosen independent of a, then we will say that $\mathfrak{A}$ has bounded nilindex, or nilindex t if t is the smallest such integer. If the powers of every a in $\mathfrak{A}$ are independent of the association of factors, i.e., if every single element of $\mathfrak{A}$ generates an associative subalgebra, then $\mathfrak{A}$ is called power-associative. Every element a of $\mathfrak{A}$ gives rise, by left and by right multiplication, to linear transformations of $\mathfrak{A}$ considered as a vector space over F. The transformations defined by $x \longrightarrow xa$ and $x \longrightarrow ax$ for all x in $\mathfrak{A}$ are generally denoted, respectively, by R_a and L_a. As the algebras considered in this paper will all be commutative, we shall in fact use only R_a, and may write xR_a for xa, xR_a^2 for $(xa)a$, and so forth.

The principal purpose of this paper is to demonstrate 1) if $\mathfrak{A}$ is a commutative nilalgebra of characteristic zero and bounded nilindex t, then $R_a^{2t-3} = 0$ for all a in $\mathfrak{A}$, and 2) if $\mathfrak{A}$ is a commutative, power-associative algebra of characteristic zero, and if for some a in $\mathfrak{A}$, $a^t = 0$, then the algebra generated by $R_a, R_{a^2}, \cdots$, is nilpotent of index not more than $2(t-1)^2 + 1$; in particular, R_a is nilpotent. (The algebra generated by all R_{a^k} is in fact generated by R_a and R_{a^2} alone, a result due to Albert [1].) The assumption of characteristic zero is not strictly necessary in the statements of the main propositions. Indeed, the assumption that the characteristic is sufficiently high compared to t will serve.

The material of this paper is drawn in part from the author's doctoral dissertation, University of Chicago, 1951, and arises from an attempt to answer certain questions raised by Albert in [1]. The author wishes to express his thanks to Professor Albert for his kindly advice and assistance, and to him and to Professor L. J. Paige for their careful reading of the manuscript.

Linearization.. In what follows, systematic use will be made of certain lemmas on linearization, some of them well-known. For completeness, proofs of the lemmas needed are given in this section.

Let V and W be vector spaces over a field F. The direct sum of V with itself n times will be denoted by V^n, and functions f^n from V^n to W (which for the moment we do *not* consider to be multilinear) will be said to have "grade" n,

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