ELEMENTS OF A THEORY OF INTRINSIC FUNCTIONS ON ALGEBRAS

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1. **Introduction.** There is a substantial amount of literature dealing with various approaches to an extension of concepts of classical function theory to finite dimensional linear associative algebras with identity over the real or complex field. (See [27] for an extensive bibliography through 1939.) If $\mathfrak{A}$ is such an algebra, with basis elements $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$, and if $\xi = x_1\epsilon_1 + \cdots + x_n\epsilon_n$ is an element of $\mathfrak{A}$, then a function $F(\xi)$ on $\mathfrak{A}$ to $\mathfrak{A}$ has the form

$$F(\xi) = \sum_{i=1}^{n} f_i(x_1, \cdots, x_n)\epsilon_i,$$

where the f_i are ordinary functions of the coordinate variables $(x_1, \cdots, x_n)$. If no further hypotheses, other than continuity or differentiability conditions, are placed on $F(\xi)$, then the resulting theory is simply a theory of mappings of the vector space $\mathfrak{A}$ into itself. More of the algebraic character of $\mathfrak{A}$ can be injected into $F(\xi)$ by requiring that the differential of $F(\xi)$ be expressible as a linear homogeneous polynomial in $d\xi = \sum dx_i\epsilon_i$, viz.,

$$dF(\xi) = \sum \frac{\partial f_i}{\partial x_j} dx_j\epsilon_i = \sum \rho_i \, d\xi\theta_i ,$$

where ρ_i, θ_i are elements of $\mathfrak{A}$ independent of $d\xi$. This is the Hausdorff differentiability condition. However, it turns out [21] that this condition in fact places comparatively little additional restriction on $F(\xi)$, especially for semi-simple algebras. (A singular exception is the case where $\mathfrak{A}$ is the algebra of complex numbers over the real numbers, in which case the condition of Hausdorff differentiability results in the Cauchy-Riemann condition.) As a consequence, the function theory developed from these, or similar, hypotheses has resulted in a substantially less rich theory than occurs in the classical case of functions on the algebra of complex numbers.

It seems reasonable to take the view that one reason for the relatively undistinguished character of the foregoing function theories is that the functions so defined are not sufficiently intimately bound up with the ring features of the algebra. One algebraically natural avenue of incorporating more algebraic structure into the function theory is treated here. Briefly, the motivation for this avenue is the following. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be isomorphic algebras over the same field $\mathfrak{F}$. If $F(\xi)$ is defined on $\mathfrak{A}$ as above, and if $\epsilon_i \to \epsilon_i'$ are isomorphic bases

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