THE COMPOSITION OF GRAPHS

By Gert Sabidussi

1. **Introduction.** In a recent paper [1] Harary introduces a new binary operation on graphs called *composition* (or *lexicographic multiplication*) (see [1; 31], or Definition 2, below), and discusses the automorphism group of the composition in terms of the automorphism groups of the composants. It is stated in [1: 32] that $G(X \circ Y) = G(X) \circ G(Y)$ (see [1; 30], or Definition 1, below) if and only if at most one of X and Y is complete. This is however incorrect, the condition being only necessary but not sufficient. It is the purpose of this note to give the correct statement of Harary's theorem, and to enlarge slightly the class of graphs under consideration.

By a *graph* X we mean a set $V(X)$ (the set of *vertices* of X) together with a set $E(X)$ (the set of *edges* of X) of unordered pairs of *distinct* elements of $V(X)$. Unordered pairs will be denoted by brackets. If A is a set, $| A |$ denotes the cardinal of A. By $| X |$ we mean $| V(X) |$. For $x \in V(X)$ we put $V(X; x) = \{y \in V(X) \mid [x, y] \in E(X)\}$. $d(X; x) = | V(X; x) |$ is the *degree* of x in X. A graph X is *almost locally finite* if for any two distinct vertices x, y, $V(X; x) \cap V(X; y)$ is finite. By the *complement* of a graph X we mean the graph X' given by $V(X') = V(X)$, $E(X') = \{[x, y] \mid x, y \in V(X'), x \neq y, [x, y] \notin E(X)\}$. An *automorphism* of X is a one-one function ϕ of $V(X)$ onto $V(X)$ such that $[\phi x, \phi y] \in E(X)$ if and only if $[x, y] \in E(X)$. By $G(X)$ we denote the automorphism group of X. Note that $G(X) = G(X')$.

Definition 1. Let A and B be sets, G and H groups of one-one functions of A onto itself, and B onto itself, respectively. Define $G \circ H$ (the *composition* of G and H) to be the group of all one-one functions f of $A \times B$ onto itself for which there exist $g \in G$ and $h_a \in H$, $a \in A$, such that

$$f(a, b) = (ga, h_{ga}b)$$

for all $(a, b) \in A \times B$.

Definition 2. Let X and Y be graphs. By the *lexicographic product* (or *composition*) $X \circ Y$ we mean the graph given by

$$V(X \circ Y) = V(X) \times V(Y),$$

$$E(X \circ Y) = \{[(x, y), (x', y')] \mid [x, x'] \in E(X),$$

or

$$x = x' \quad \text{and} \quad [y, y'] \in E(Y)\}.$$

It is easily verified that $X \circ (Y \circ Z) \cong (X \circ Y) \circ Z$, and that $(X \circ Y)' = X' \circ Y'$. Idempotency is possible, e.g. if C_n is the complete n-graph, and n is infinite, then $C_n \circ C_n \cong C_n$.

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