INEQUALITIES AMONG SOME REAL MODULAR FUNCTIONS

By Peter Flor

Let α be a real irrational number. The properties of the linear function $x - \alpha y$ with integral arguments depend mainly on the functions

$$I(\alpha) = \liminf_{t\to\infty} \quad \min_{0 < |y| \leq t,\ x \text{ and } y \text{ integers}} t\,|x - \alpha y|$$

and

$$S(\alpha) = \limsup_{t\to\infty} \quad \min_{0 < |y| \leq t,\ x \text{ and } y \text{ integers.}} t\,|x - \alpha y|$$

(Koksma [1; 37]).

Let α be developed into a regular continued fraction, $\alpha = (a_0, a_1, a_2, \cdots)$. Put $\alpha_n = (a_n, a_{n+1}, a_{n+2}, \cdots)$, $\beta_n = (a_{n-1}, a_{n-2}, \cdots, a_1)$. Then it can be shown easily (Morimoto [2]) that

$$I(\alpha) = \liminf_{n\to\infty} \beta_n/(\alpha_n\beta_n + 1)$$

and

$$S(\alpha) = \limsup_{n\to\infty} \alpha_n\beta_n/(\alpha_n\beta_n + 1).$$

Let us define $k(\alpha) = \limsup_{n\to\infty} a_n$. With this notation, we can easily deduce from Morimoto's formulas the following

Theorem I. *Let α be any real irrational number. Then the following statements are equivalent:*

(a) $$k(\alpha) = \infty.$$

(b) $$I(\alpha) = 0.$$

(c) $$S(\alpha) = 1.$$

The *main* purpose of this paper is to prove some inequalities relating these functions $k(\alpha)$, $I(\alpha)$, and $S(\alpha)$; more exactly speaking, to give upper and lower estimates for each of these functions in terms of each of the other two. These inequalities will serve to prove

Theorem II. *Let $\{\gamma_i\}$ denote any sequence of real irrational numbers. Then the following statements are equivalent:*

Received November 11, 1958. The results of this paper constitute part of the dissertation presented by the author for the degree of Doctor of Philosophy at the University of Vienna, Austria. The author wishes to express his gratitude to Professor E. Hlawka under whose direction the dissertation was written.