INVESTIGATION CONCERNING POSITIVE DEFINITE CONTINUED FRACTIONS

The author dedicates this paper to his teacher, H. S. Wall, on his fifty-sixth anniversary.

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1. **Introduction.** The object of this investigation is a proper extension of the theory of positive definite continued fractions. Positive definite continued fractions are generated by an infinite sequence of linear-fractional transformations of the form

$$t_p(U) = \frac{A_{p-1}^2}{Z_p - B_p - U} \qquad (p = 1, 2, \cdots) \tag{1.1}$$

where the complex-number sequences A and B satisfy the condition that—for every positive integer n—

$$\sum_{p=1}^{n+1} \operatorname{Im}(Z_p - B_p) \mid x_p \mid^2 - \sum_{p=1}^{n} \operatorname{Im}(A_p)[x_p x_{p+1}^* + x_p^* x_{p+1}] \geq 0 \tag{1.2}$$

for each sequence Z of complex numbers with positive imaginary part and each sequence x of complex numbers. Essential parts of that theory are: (I) consideration of the condition—equivalent to (1.2)—that there exist a sequence $\{g_0, g_1, \cdots\}$ such that

$$\left.\begin{aligned} &\mid A_p^2 \mid - \operatorname{Re}(A_p^2) \leq 2(1 - g_{p-1}) g_p b_p b_{p+1}, \\ &0 \leq g_{p-1} \leq 1, \quad \text{and} \quad 0 \leq b_p = \operatorname{Im}(-B_p) \end{aligned}\right\} \qquad (p = 1, 2, \cdots); \tag{1.3}$$

(II) determination that if $\operatorname{Im}(U) \leq (1 - g_p) b_{p+1}$ then

$$\operatorname{Im} t_p(U) \leq (1 - g_{p-1}) b_p \quad \text{for} \quad \operatorname{Im}(Z_p) \geq 0, \tag{1.4}$$

and if $\operatorname{Im}(U) \leq (1 - g_0) b_1$ and $0 < y_1 = \operatorname{Im}(Z_1)$, then

$$\left| t_1(U) + \frac{i}{2} \frac{A_0^2}{y_1 + g_0 b_1} \right| \leq \frac{1}{2} \frac{\mid A_0^2 \mid}{y_1 + g_0 b_1}; \tag{1.5}$$

and (III) consideration of the *nest of circles* $\{K_1(Z), K_2(Z), \cdots\}$ in the complex plane, where—with the suppositions that $\operatorname{Im}(Z_1) > 0$ and $\operatorname{Im}(Z_p) \geq 0$ ($p = 2, 3, \cdots$)—$K_1(Z)$ is the image under t_1 of the half-plane $\operatorname{Im}(U) \leq (1 - g_0) b_1$ and for each positive integer n $K_{n+1}(Z)$ is the image of the half-plane $\operatorname{Im}(U) \leq (1 - g_n) b_{n+1}$ under the composite mapping $t_1 \cdots t_{n+1}$.

The theory which I have indicated here had its beginning in the fundamental papers of Scott and Wall [5], Paydon and Wall [4], Hellinger and Wall [2],

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