THE GREATEST DISTANCE BETWEEN TWO CHARACTERISTIC ROOTS OF A MATRIX

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1. **Introduction.** In two recent papers L. Mirsky [3],[4] has obtained upper and lower bounds for the greatest distance between two characteristic roots of a matrix. In particular he proved the following theorems.

THEOREM I. [4, Theorem 2] *Let $A = (a_{ij})$ be a square matrix of order n with only real roots. Let $f(x) = x^n - c_1x^{n-1} + c_2x^{n-2} - \cdots + (-1)^n c_n = 0$ be its characteristic equation and $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_n$ its characteristic roots. If we set*

$$K_2 = \{2(1 - 1/n)c_1^2 - 4c_2\}^{\frac{1}{2}}, \tag{1}$$

then

$$\omega_1 - \omega_n \leq K_2 . \tag{2}$$

It may be remarked that this theorem is not new but is actually a theorem of J. v. Sz. Nagy [5; 42] on algebraic equations.

THEOREM II. [3, Theorem 2] *If A is an Hermitian matrix, then*

$$\omega_1 - \omega_n \geq \max_{\kappa \neq \lambda} \{(a_{\kappa\kappa} - a_{\lambda\lambda})^2 + 4 \mid a_{\kappa\lambda} \mid^2\}^{\frac{1}{2}}.$$

THEOREM III. [3, Theorem 4] *Let $A = (a_{ij})$ be a normal matrix of order n with characteristic roots $\omega_1, \omega_2, \cdots, \omega_n$. Then*

$$\max \mid \omega_i - \omega_j \mid \geq (1/\sqrt{2}) \max_{\kappa \neq \lambda} \{\mid a_{\kappa\kappa} - a_{\lambda\lambda} \mid^2 + \mid (a_{\kappa\kappa} - a_{\lambda\lambda})^2 + 4a_{\kappa\lambda}a_{\lambda\kappa} \mid + 2 \mid a_{\kappa\lambda} \mid^2 + 2 \mid a_{\lambda\kappa} \mid^2\}^{\frac{1}{2}}.$$

In this paper we extend Theorem I. Let A be a matrix with only real roots and let K_2 be defined by (1). Then

$$\begin{aligned} K_2 &\geq \max \mid \omega_i - \omega_j \mid \geq (2/n)^{\frac{1}{2}} K_2 , \qquad && n \text{ even}, \\ K_2 &\geq \max \mid \omega_i - \omega_j \mid \geq \{2n/(n^2 - 1)\}^{\frac{1}{2}} K_2 , && n \text{ odd}. \end{aligned} \tag{3}$$

The lower bounds are often better than Theorem II even if this theorem is written in a somewhat stronger form. These bounds are obtained from a theorem of T. Popoviciu [7] on algebraic equations with only real roots. We will give proofs for these results which are simpler than those of v. Sz. Nagy, Mirsky, and Popoviciu. Moreover, K_2 can be replaced in (3) by functions K_{2s} which depend not only upon the first two but upon the first $2s$ elementary symmetric functions of the roots, for $2 \leq s \leq [n/2]$.

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