A THEOREM ON SEMI-CONTINUOUS SET-VALUED FUNCTIONS

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1. **Introduction.** The purpose of this paper is to prove a rather curious result about semi-continuous set-valued functions (Theorem 1.1), and to derive some applications to open and closed point-valued functions.

Recall that if X and Y are topological spaces, and if 2^Y denotes the space of non-empty subsets of Y, then a function $\phi: X \to 2^Y$ is called *lower* (respectively *upper*) *semi-continuous* if

$$\{x \in X \mid \phi(x) \cap U \neq \emptyset\} \qquad \text{(respectively} \qquad \{x \in X \mid \phi(x) \subset U\})$$

is open in X for every open $U \subset Y$. An important example is the case where ϕ is of the form $\phi = u^{-1}$ for some function u from Y onto X; in this case, ϕ is lower (respectively upper) semi-continuous if and only if u is open (respectively closed).

THEOREM 1.1. *Let X be paracompact, Y a metric space, and $\phi: X \to 2^Y$ lower semi-continuous with each $\phi(x)$ complete. Then there exist $\psi: X \to 2^Y$ and $\theta: X \to 2^Y$ such that*

(a) *$\psi(x) \subset \theta(x) \subset \phi(x)$ for all $x \in X$,*
(b) *$\psi(x)$ and $\theta(x)$ are compact for all $x \in X$,*
(c) *ψ is lower semi-continuous,*
(d) *θ is upper semi-continuous.*

COROLLARY 1.2. *Let E be a metric space, F paracompact (Hausdorff will suffice in (b)), and $f: E \to F$ open and onto with $f^{-1}(y)$ complete for every $y \in F$. Then*

(a) *There exist subsets $E'' \subset E' \subset E$ such that $f(E'') = f(E') = F$, $f \mid E'$ is closed, $f \mid E''$ is open, and for each $y \in F$ the sets $(f \mid E')^{-1}(y)$ and $(f \mid E'')^{-1}(y)$ are compact.*

(b) *If $B \subset F$ is compact, then there exists a compact $A \subset E$ such that $f(A) = B$.*

(c) *If f is continuous, then F is metrizable.*

Observe that Corollary 1.2. (b) generalizes a result of Bourbaki [1, §2, Proposition 18], where f is continuous, and E itself—rather than just the sets $f^{-1}(y)$—is required to be complete.

It should be noted that the roles of open and closed maps in Corollary 1.2 (a) cannot be reversed. For instance, the map $f: [0,3] \to [0,2]$, defined by $f(x) = x$ if $0 \leq x \leq 1$, $f(x) = 1$ if $1 \leq x \leq 2$, and $f(x) = x - 1$ if $2 \leq x \leq 3$, is continuous and closed, but there exists no $A \subset [0, 3]$ such that $f(A) = [0, 2]$ and $f \mid A$ is open.

The proof of Theorem 1.1 will be found in §§2 and 3; the lemma in §2 may

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