CONCERNING THE HAUSDORFF INCLUSION PROBLEM

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1. **Introduction.** Let $Q[0, 1]$ denote the class of all real-valued functions ϕ defined on the set of real numbers such that (1) $\phi(x) = 0(x \leq 0)$ and $\phi(x) = \phi(1)(x > 1)$, (2) ϕ is quasicontinuous on $[0, 1]$ and (3) $\phi(x) = \frac{1}{2}[\phi(x +) + \phi(x -)]$ $(0 < x < 1)$. If $\phi \varepsilon Q[0, 1]$, condition (2) means that ϕ possesses discontinuities of the first kind only while condition (3) states that ϕ is normalized at $x(0 < x < 1)$. A sequence $\{a_n\}$ of real numbers is a *Q-sequence* means there exists a mass function $\phi \varepsilon Q[0, 1]$ such that

$$(1.1) \qquad a_n = \int_0^1 x^n \, d\phi(x) \qquad (n = 0, 1, 2, \cdots).$$

A Q-sequence $\{a_n\}$ is called a *regular moment sequence* if ϕ is a *regular mass function*, i.e., ϕ is of bounded variation, $\phi(0 +) = 0$ and $\phi(1) = 1$. Thus, the Q-sequences constitute a class of moment sequences which contains the class of regular moment sequences as a proper subset.

If a denotes the number sequence $\{a_n\}$, the statement that the transformation H_a from the set S of all real number sequences into a subset of S is *the Hausdorff transformation determined by the number sequence* a means that, if $s \varepsilon S$ and $t = H_a s$, then

$$t_n = \sum_{p=0}^{n} \binom{n}{p} \Delta^{n-p} a_p \cdot s_p \qquad (n = 0, 1, 2, \cdots),$$

where $\Delta^0 a_p = a_p$ and $\Delta^k a_p = \Delta^{k-1} a_p - \Delta^{k-1} a_{p+1} (k = 1, 2, 3, \cdots)$. If a is a Q-sequence with mass function ϕ, then $H_a = H^\phi$, the *Hausdorff transformation determined* by ϕ, and if $t = H^\phi s$,

$$t_n = \int_0^1 \sum_{p=0}^{n} \binom{n}{p} x^p (1 - x)^{n-p} s_p \, d\phi(x) \qquad (n = 0, 1, 2, \cdots).$$

If a is a regular moment sequence, then $H_a = H^\phi$ is called a *regular Hausdorff transformation.*

Let $\{a_n\}$ and $\{b_n\}$ be two Q-sequences. H_a *includes* H_b means that if s is a real number sequence and $\lim_{n \to \infty} (H_b s)_n$ exists and is the number k, then $\lim_{n \to \infty} (H_a s)_n = k$. Hausdorff showed that, if no term of $\{b_n\}$ is 0, then H_a includes H_b if and only if $\{a_n / b_n\}$ is a regular moment sequence. Other formulations of this "Hausdorff inclusion problem" have been given by Hille and Tamarkin [3] and by Garabedian, Hille and Wall [1] in case each of $\{a_n\}$ and

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