CHARACTERISTICALLY NILPOTENT LIE ALGEBRAS

By G. Leger and S. Tôgô

In a recent paper [4], Dixmier and Lister have given an example of a Lie algebra all of whose derivations are nilpotent and then distinguished a subclass of nilpotent Lie algebras as characteristically nilpotent. Namely, let L be a Lie algebra and let $D(L)$ be its derivation algebra i.e. the Lie algebra of all derivations of L. Let $L^{[1]} = D(L)L = \{\sum D_i x_i \mid x_i \in L, D_i \in D(L)\}$ and define $L^{[k+1]} = D(L)L^{[k]}$ inductively. L is called *characteristically nilpotent* if there exists an integer n such that $L^{[n]} = (0)$.

It is the purpose of this paper to study characteristically nilpotent Lie algebras and to present some results on their structure.

1. Let L be a Lie algebra over a field F. Let K be any extension field of F and L^K be the Lie algebra obtained by extending the ground field F to K. Every derivation of L may be also considered as the derivation of L^K to which it extends and then we have $D(L^K) = D(L)^K$. It is seen at once from this fact that L^K is characteristically nilpotent if and only if L is characteristically nilpotent. Further, if L is characteristically nilpotent, it is evident that all derivations of L are nilpotent. The converse of this fact is also true. Indeed, if all derivations of L are nilpotent, then Engel's theorem says that the intersection of all associative algebras of linear transformations of L containing $D(L)$ is a nilpotent associative algebra, and therefore L is characteristically nilpotent.

Lemma 1. *If L is characteristically nilpotent, then*
(1) *the center of L is contained in* $[L, L]$;
(2) $L^3 \neq (0)$.

Proof. If (1) or (2) is not satisfied, then it is easy to construct a non-nilpotent derivation of L, and L is not characteristically nilpotent.

Lemma 2. *Let L be a nilpotent Lie algebra. If L is the direct sum of two non-zero ideals one of which is central, then $D(L)$ is not nilpotent.*

Proof. Let $L = L_1 + Z$ be the direct sum of an ideal L_1 and a central ideal Z. Take an element $x \neq 0$ in Z and let U be a complementary subspace of (x) in Z. Since L_1 is nilpotent, there exists an element $y \neq 0$ in the center of L_1. Now define two derivations of L in the following way:

$$DL_1 = (0), \qquad Dx = y \quad \text{and} \quad DU = (0);$$

$$D'L_1 = (0), \qquad D'x = x \quad \text{and} \quad D'U = (0).$$

Then we have $[D, D'] = D$, from which it follows that $D(L)$ is not nilpotent.

Received December 27, 1958. The work of the second author was supported by the National Science Foundation.