POSITIVE DEFINITE SEQUENCES AND ABSOLUTELY MONOTONIC FUNCTIONS

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Let F be a real function defined on the segment $(-1, 1)$, and consider the following four properties which F may satisfy:

(A) If $-1 < a_n < 1$ $(n = 0, \pm1, \pm2, \cdots)$ and if $\{a_n\}$ is a positive definite sequence, then the sequence $\{F(a_n)\}$ is also positive definite.

(B) If $-1 < a_n < 1$ $(n = 0, \pm1, \pm2, \cdots)$ and if $\{a_n\}$ is a sequence of Fourier-Stieltjes coefficients, then $\{F(a_n)\}$ is also a sequence of Fourier-Stieltjes coefficients.

(C) If $-1 < a_{ij} < 1$ and if (a_{ij}) is a positive definite in $n \times n$ matrix, then the matrix $(F(a_{ij}))$ is also positive definite.

(D) $F(x) = \sum_0^\infty c_n x^n$ $(-1 < x < 1)$, with $c_n \geq 0$ $(n = 0, 1, 2, \cdots)$.

Let us recall that $\{a_n\}$ is said to be a positive definite sequence if

$$(1) \qquad \sum_{i,j=1}^n a_{i-j} x_i \bar{x}_j \geq 0$$

for every n and for every complex vector $(x_1, \cdots, x_n)$. Thus (A) is equivalent to

(A′) If (a_{i-j}) $(i, j = 1, \cdots, n)$ is a positive definite $n \times n$ matrix for every n, then so is the matrix $(F(a_{i-j}))$.

Evidently (A) is thus a weaker requirement than (C). In 1942, Schoenberg [3; 97] proved that every continuous F which satisfies (C) is of the form (D). In the present paper it is shown that it is possible to weaken (C) to (A), and to drop the assumption of continuity:

THEOREM I. *If F satisfies* (A), *then F is of the form* (D).

(The converse is trivial, by (2) below.) Since $\{a + b \cos n\,\alpha\}$ is a positive definite sequence whenever a and b are positive, Theorem I is a corollary of Theorem IV below, whose hypotheses are even weaker.

The connection between conditions (A) and (B) is that the positive definite sequences are precisely those which have a representation

$$(2) \qquad a_n = \int_{-\pi}^{\pi} e^{-in\theta}\, d\mu(\theta) \qquad (n = 0, \pm1, \pm2, \cdots),$$

where μ is a positive measure. Thus (A) is a condition similar to (B), except that in (A) F is required to operate on the positive cone of the measure algebra on the circle, and not on the whole algebra. It has recently been proved [1] that F can be extended to an entire function in the complex plane, if (B) holds.