THE ANALOGUE FOR MAXIMALLY CONVERGENT POLYNOMIALS OF JENTZSCH'S THEOREM

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Although sequences of rational and analytic functions of the complex variable have been studied in some detail [3], [4], [5] regarding geometric degree of convergence, and convergent sequences of analytic functions have been basically studied [6] with reference to their zeros, no adequate applications to maximally convergent sequences of polynomials as such have been made. In particular there has been no special study of the analogue of Jentzsch's theorem, that every point of the circle of convergence of a Taylor development is a limit point of zeros of the partial sums. The object of the present paper is to investigate specifically the zeros of maximally convergent sequences of polynomials, including in detail some illuminating special expansions.

1. **Exact harmonic majorants.** If E is a closed bounded point set of the $z(=x+iy)$ $-$plane whose complement K is connected, and is regular in the sense that Green's function $G(x, y)$ exists for K with pole at infinity, we denote generically by C_σ the locus $G(x, y) = \log \sigma$ (>0) in K, and by E_σ the subset $0 < G(x, y) < \log \sigma$ of K. If a function $f(z)$ is single-valued and analytic on E, there is a largest σ finite or infinite, say $\sigma = \rho$, such that $f(z)$ is single-valued and analytic on $E + E_\rho$. There exist [3, §4.7] polynomials $p_n(z)$ of respective degrees $n = 1, 2, \cdots$ such that we have

$$\limsup_{n\to\infty} [\max | f(z) - p_n(z) |, z \text{ on } E]^{1/n} = 1/\rho , \tag{1}$$

but there exist no polynomials $p_n(z)$ of respective degrees n such that the first member of (1) is less than $1/\rho$. A sequence $p_n(z)$ satisfying (1) is said to converge *maximally* to $f(z)$ on E; it converges to $f(z)$ throughout $E_\rho + E$, uniformly on any closed bounded set in $E_\rho + E$. In considering the analogue of Jentzsch's theorem, we shall be concerned with the zeros of the $p_n(z)$ on and near C_ρ, for n arbitrarily large. If the sequence $p_n(z)$ converges maximally to $f(z)$ on E, it also converges [3, §4.7] maximally to $f(z)$ on every $C_\sigma + E_\sigma + E$, $1 < \sigma < \rho$, so (in the present study) it is no loss of generality to assume that E is bounded by a finite number of mutually exterior Jordan curves. For arbitrary $f(z)$ analytic at $z = 0$, the partial sums of the Maclaurin series converge maximally to $f(z)$ on every closed circular disc $| z | \leq r$ interior to the circle of convergence.

In the case $\rho = \infty$ with arbitrary E, the function $f(z)$ is entire, and if $f(z) \not\equiv 0$, the (finite) points of the plane that are limit points of zeros of a maximally

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