SOME ARITHMETIC PROPERTIES OF A SPECIAL SEQUENCE OF POLYNOMIALS

To Alfred Brauer on his sixty-fifth birthday

By L. Carlitz

1. Kelisky [2] has defined a set of integers T_n by means of

$$\sum_0^\infty T_n \frac{x^n}{n!} = e^{\arctan x} \tag{1.1}$$

and obtained a number of properties of these numbers, in particular the following interesting arithmetic properties.

If p is an odd prime, then

$$T_p \equiv \begin{cases} 0 \pmod p & (p = 4n + 1) \\ 2 \pmod p & (p = 4n + 3); \end{cases} \tag{1.2}$$

if p is a prime of the form $4n + 1$, $m \geq 1$, $[m/p] = r$, then

$$T_m \equiv 0 \pmod{p^r}. \tag{1.3}$$

In place of (1.1) there is the equivalent definition

$$\left(\frac{1 + ix}{1 - ix}\right)^{-i/2} = \sum_0^\infty T_n \frac{x^n}{n!}, \tag{1.4}$$

which suggests the generalization

$$\left(\frac{1 + ix}{1 - ix}\right)^{-iz/2} = \exp(z \arctan x) = \sum_0^\infty T_n(z) \frac{x^n}{n!}. \tag{1.5}$$

The polynomial $T_n(z)$ has integral coefficients; indeed if we put

$$\frac{1}{k!} (\arctan x)^k = \sum_{n=k}^\infty A_n^{(k)} \frac{x^n}{n!} \tag{1.6}$$

for $k = 0, 1, 2, \cdots$, then by known properties of Hurwitz series, the $A_n^{(k)}$ are integers and comparison of (1.5) and (1.6) yields

$$T_n(z) = \sum_{k=0}^n A_n^{(k)} z^k. \tag{1.7}$$

It is of course evident from (1.1) and (1.5) that

$$T_n(1) = T_n. \tag{1.8}$$

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