FUNCTIONS WITH POSITIVE REAL PART AND SCHLICHT FUNCTIONS

BY S. D. BERNARDI

1. **Introduction.** The Noshiro-Warschawski theorem [6, Theorem 12; 151] and [10, Lemma 1; 312], which is a generalization of a result by J. Wolff [11], states

(1.1) *If* Re $p(z) > 0$ *in a convex domain* D_c , *then* $\int_0^z p(t)\, dt$ *is univalent (schlicht) in* D_c .

The purpose of this paper is to generalize this result as is indicated in Theorem 1, using, with a slight modification, the method employed in [10]. Corollary 1 contains the result (1.1) as a special case and is the source of most of the results presented in the remainder of the paper. The following terminology will be employed. Let δ denote the interior of the unit circle $|z| = 1$. We shall say that $f(z) \subset (S)$ if $f(z) = z + a_2 z^2 + \cdots$ is regular and univalent in δ. The notation $f(z) \subset (ST)$ indicates that $f(z) \subset (S)$ and maps δ onto a region which is starlike with respect to the origin. Similarly, $f(z) \subset (SC)$ signifies that $f(z) \subset (S)$ and maps δ onto a convex region. Schlicht functions with convex maps are denoted by $f_c(z) = z + \cdots$, while those with starlike maps are denoted by $f_t(z) = z + \cdots$. The principal theorem to be proven is

THEOREM 1. *Let* $w = f(z) \subset (S)$ *map* δ *onto a region* D. *Let* D_c *denote any convex region containing* D, $D_c \geq D$. *Suppose the inverse function* $f^{-1}(w)$ *is regular in* D_c *and maps* D_c *onto a region* D_{-c} . *Let* $p(z)$ *be regular and* Re $p(z) > 0$ *in* D_{-c} . *Then* $\int_0^z p(t) f'(t)\, dt$ *is regular and univalent in* D_{-c} .

(1.2) COROLLARY 1. *Let* $f(z) \subset (SC)$. *Let* $p(z) = 1 + p_1 z + \cdots$ *be regular and* Re $p(z) > 0$ *in* δ. *Then* $\int_0^z p(t) f_c'(t)\, dt \subset (S)$.

2. **Proof of Theorem 1.** We note that since $f(z)$ is univalent in δ, it follows that $D_{-c} \geq \delta$. Make the change of variables $u = f(\zeta)$ and obtain

$$F(z) = \int_0^z p(\zeta) f'(\zeta)\, d\zeta = \int_0^{f(z)} p[f^{-1}(u)]\, du.$$

Let z_1 , z_2 be any two points of δ, and let $f(z_2) = f(z_1) + Le^{i\alpha}$. Integrating along the straight segment joining $f(z_1)$ and $f(z_2)$ we have $u = f(z_1) + te^{i\alpha}$, $0 \leq t \leq L$, so that

$$| F(z_2) - F(z_1) | = \left| \int_{f(z_1)}^{f(z_2)} p[f^{-1}(u)]\, du \right| = \left| \int_0^L p[f^{-1}(f(z_1) + te^{i\alpha})] e^{i\alpha}\, dt \right|.$$

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