A PROPERTY OF ENTIRE FUNCTIONS OF SMALL ORDER AND FINITE TYPE

By Q. I. Rahman

The object of this note is to prove the following

THEOREM. *Let $f(z)$ be an entire function of order ρ $(0 < \rho < 2)$ and finite type. If $f(0) = 1$, then*

$$\int_R^S \frac{\log |f(z)|}{|z|^{1+\rho}} \, d|z| \tag{A}$$

is bounded as R and S tend to infinity with $R < S$ and $S/R = O(1)$ and z varies along any path joining R and S.

The above theorem is of the same general nature as the Faber-Pólya theorem [3, Theorem 13], although neither includes the other.

Proof of the theorem. Clearly (A) is bounded above. To show that it is bounded below it is sufficient to prove that

$$\int_R^S \frac{\log |f(z)f(-z)|}{|z|^{1+\rho}} \, d|z|$$

is bounded below. Let $z = re^{i\theta}$ and

$$f(z) = e^{cz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right);$$

then

$$f(z)f(-z) = g(z^2) = g(r^2 e^{2i\theta}) = \prod_{n=I}^{\infty} \left(1 - \frac{z^2}{z_n^2}\right).$$

Hence we may consider

$$\int_R^S \frac{\log |g(r^2 e^{2i\theta})|}{r^{1+\rho}} \, dr = \frac{1}{2} \int_{R^2}^{S^2} \frac{\log |g(te^{2i\theta})|}{t^{1+\rho/2}} \, dt.$$

$g(z)$ is of mean type, order $\rho/2$. So is

$$G(z) = \prod_{n=I}^{\infty} \left(1 + \frac{z}{r_n^2}\right),$$

and

$$|G(-r)| \leq |g(re^{i\theta})|.$$

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