AN APPROXIMATE SOLUTION OF AN IMPROPER BOUNDARY VALUE PROBLEM

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1. **Introduction.** Hadamard called those boundary value problems whose solutions do not depend continuously on the boundary data "improperly posed." He was aware that the data which are used in practice are usually inexact, due, for instance, to the use of rational approximations to irrational numbers. Thus a solution to such a boundary value problem is as likely to reflect the influence of small random oscillations of the error in the boundary data as it is to reflect the influence of the correct values which the data are supposed to approximate. Recently, however, mathematicians have seen that these apparently improperly posed problems are of interest and that the correct solution can in many cases be approximated to any desired degree of accuracy provided that sufficient precautions are taken in choosing the method of approximation and provided that the magnitude of the error in the data can be controlled [1; 4–10, 12, 13].

In this paper we give a new solution of an improperly posed problem first solved by Kreisel [8]. His method of solution is to seek smoothed solutions of an integral equation; our method is to seek a normal family of solutions of the corresponding differential equation. Since a normal family need not be smooth in Kreisel's sense and a family of smooth functions need not be normal, the two methods would appear to be quite unrelated.

Kreisel gives no explicit estimate of the error in his method, and such an estimate is impossible to obtain from the method of proof used here because of our use of normal families of functions. However, numerical experiments with the two methods have indicated that the method presented here gives a considerably better approximation to the true solution.

2. **Statement of the problem.** Let u be a function about which nothing is known except that on the half-plane $y \geq 0$ it is bounded and harmonic and that on a certain line $y = c$ where $c > 0$, an arbitrarily good approximation to the values of u can be obtained. By this we mean that, given any $\epsilon > 0$, we can find some function F_ϵ such that, for $-\infty < x < \infty$,

$$| F_\epsilon(x) - u(x, c) | < \epsilon. \tag{1}$$

In applications F_ϵ is frequently an interpolating function determined by measurements of the values of u at a finite number of points on the line $y = c$. We will refer to the values of F_ϵ as boundary data, even when the line $y = c$ on which the data are given is in the interior of the region in which we are interested.

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