LOCAL CONTRACTIONS AND THE SIZE OF A COMPACT METRIC SPACE

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The intuitive notion that a compact metric space X has "size" is borne out by the theorems:

A. If $f:X \to X$ is an isometry, then f is onto.

B. If $f:X \to$ X is onto and increases no distances, then f is an isometry.

C. If $f:X \to Y$ increases no distances, then the Hausdorff p-measure of Y is $\leq$ that of X.

B and C can be strengthened by replacing 'isometry' with 'ϵ-isometry' and the phrase 'increases no distances' (i.e., is a contraction) with 'is an ϵ-contraction'. These last terms are to be construed as follows: $f:X \to Y$ is an ϵ-contraction, [respectively, ϵ-isometry], if and only if $\epsilon > 0$ and $\rho[f(x), f(x')] \leq$ [respectively, $=$], $\rho(x, x')$, provided $\rho(x, x') < \epsilon$.

A. Edrei [3] introduced the generalizations 'local contraction' and 'local isometry' in a paper motivated by Theorem B. For these mappings, he proves the analogue to B under various additional metrical hypotheses. That some additional hypothesis is necessary for the analogues to both A and B is shown in [6].

In §1 of the present paper, the analogue to B is proved where the additional hypotheses are purely topological. In §2 examples are given indicating the necessity of these restrictions. The length of this section seems justified by the fact that the spaces given are topologically so simple: dendrites, simple closed curves, etc. In §3 Theorem C is proved for local contractions with no additional hypothesis. Also, an analogue to A is given, where additional assumptions are made in terms of Hausdorff p-measure.

To point out the essentially metric nature of contractions, the following example is mentioned. Suppose (X, ρ) and (Y, σ) are metric spaces and f is any (continuous) map of X into Y. Define $\eta(x, x') = \max\{\rho(x, x'), \sigma[f(x), f(x')]\}$, for all $x, x' \in X$. Then η is a metric and, relative to η, f is a contraction.

Definitions and notation. All spaces considered are metric and the one symbol ρ will stand for each metric, unless otherwise indicated. If $f:X \to Y$, then f is said to be a *local contraction,* [*respectively, local isometry*], if and only if for each point $x \in X$ there exists a neighborhood N of x such that if $x' \in N$, then $\rho[f(x), f(x')] \leq$ [respectively, $=$] $\rho(x, x')$. Such a neighborhood N will be called a *neighborhood of contraction* [*isometry*] of x. A dendrite is as in Whyburn [5; 88]. If $f:X \to X$, then $f^2 =$ ff, $f^3 =$ ff^2, etc.; x is said to be of *finite period* (under f) if there exists a positive integer n such that $f^n(x) = x$, and the least such integer is called the *period* (under f) of x: a subset M of X is said to be *invariant* (under f) if and only if $f(M) \subset M$.

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